

Relative Invariants for Homogeneous Linear Differential Equations

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1. INTRODUCTION

A thorough search of the literature on relative invariants of homogeneous linear differential equations for the years 1862–1987 has shown us that the most significant contributions are a series of new results we are pleased to present in this paper. The evolution of our subject can be traced through [13–15, 32, 33, 8, 25, 26, 18, 9, 7], the bibliography in [45, pp. 193–255], and the historical information in [48, 44, 16, 49, 36, 37]. Our remarkable new identities in Theorems 4.1 and 4.2 are essential for a satisfactory theory of relative invariants. They were discovered through a careful study in Section 3 of one particular class of nonlinear differential equations.

For any integer $m \geq 2$, we suppose the coefficients $c_1(z), \dots, c_m(z)$ of the monic m th order homogeneous linear differential equation

$$y^{(m)} + \sum_{j=1}^m c_j(z) y^{(m-j)} = 0 \quad (1.1)$$

are meromorphic functions of a complex variable z on a region Ω of the complex plane. Our subject is about transformations of (1.1) under substitutions

$$y = \rho(z)v, \quad z = f(\zeta), \quad (1.2)$$

where ρ is an analytic function on some subregion U of Ω such that $\rho(z) \neq 0$, for each z in U , and $z = f(\zeta)$ is a univalent analytic function on some region V of the complex plane such that $U = f(V)$. It is customary to regard (1.2) as the composite of the change

$$y = \rho(z)v \quad (1.3)$$

of the dependent variable from y to v and the change

$$z = f(\zeta) \quad (1.4)$$

of the independent variable from z to ζ . First, for any selection of the coefficients $c_1(z), \dots, c_m(z)$ of (1.1) and any (1.2), there are unique meromorphic functions $c_1^*(z), \dots, c_m^*(z)$ of z on U such that (1.3) transforms (1.1) into

$$v^{(m)} + \sum_{j=1}^m c_j^*(z) v^{(m-j)} = 0. \quad (1.5)$$

Then, there are unique meromorphic functions $c_1^{**}(\zeta), \dots, c_m^{**}(\zeta)$ of ζ on V such that (1.4) transforms (1.5) into

$$D_\zeta^m t + \sum_{j=1}^m c_j^{**}(\zeta) D_\zeta^{m-j} t = 0, \quad (1.6)$$

where $t(\zeta) = (v \circ f)(\zeta) = v(f(\zeta))$. Throughout, we define $c_0(z)$ on Ω , $c_0^*(z)$ on U , and $c_0^{**}(\zeta)$ on V by

$$c_0(z) \equiv 1, \quad c_0^*(z) \equiv 1, \quad c_0^{**}(\zeta) \equiv 1, \quad (1.7)$$

in order to simplify later formulas such as (1.18).

For $m \geq 2$ and any meromorphic functions $c_1(z), c_2(z)$ on some region Ω , there are unique meromorphic functions $\xi_{m,1}(z), \dots, \xi_{m,m}(z)$ on Ω satisfying

$$\xi_{m,1}(z) \equiv c_1(z), \quad \xi_{m,2}(z) \equiv c_2(z), \quad (1.8)$$

and the requirement that the differential equation

$$y^{(m)} + \sum_{j=1}^m \xi_{m,j}(z) y^{(m-j)} = 0 \quad (1.9)$$

has a fundamental system of local solutions of the form

$$\{(\phi(z))^{m-1-i} (\psi(z))^i : \text{for } i = 0, 1, \dots, m-1\} \quad (1.10)$$

on some subregion of Ω . This is a consequence of Proposition 4.4 and Corollary 2.20 in which we establish the new result of basic importance that $\xi_{m,1}(z), \dots, \xi_{m,m}(z)$ are the polynomial combinations of $c_1(z), c_2(z)$, and their derivatives given by

$$\xi_{m,j}(z) \equiv a_{m-1, m-j}(z), \quad \text{for } j = 1, \dots, m, \quad (1.11)$$

where $a_{i,j}(z)$, for $i=0, 1, \dots, m-1$ and $j=-1, 0, 1, \dots, m$, is defined recursively in terms of

$$\alpha(z) \equiv \frac{2c_1(z)}{m(m-1)} \quad (1.12)$$

and

$$\beta(z) \equiv \frac{\left(6c_2(z) - 2(m-2)c_1'(z) - ((3m-1)(m-2)/m(m-1))(c_1(z))^2\right)}{(m+1)m(m-1)} \quad (1.13)$$

through

$$\begin{aligned} a_{i,-1} &\equiv 0, & \text{for } i=0, 1, \dots, m-1; & a_{0,0} \equiv 0; \\ a_{i,i+1} &\equiv 1, & \text{for } i=0, 1, \dots, m-1; \\ a_{i,j} &\equiv 0, & \text{for } i=0, 1, \dots, m-2 \text{ and } i+2 \leq j \leq m; \\ a_{1,0} &\equiv (m-1)\beta; & a_{1,1} &\equiv \alpha; \\ a_{i+1,j} &\equiv a_{i,j-1} + (a_{i,j})' + (i+1)\alpha a_{i,j} + (i+1)(m-1-i)\beta a_{i-1,j}, \\ & & \text{for } i=1, 2, \dots, m-2 \text{ and } 0 \leq j \leq i+1. \end{aligned} \quad (1.14)$$

Assuming the coefficients $c_1(z), \dots, c_m(z)$ of (1.1) are unrestricted, we shall specify relative invariants by introducing (1.12), (1.13), (1.14), (1.11),

$$B_{m,j}(z) \equiv c_j(z) - \xi_{m,j}(z), \quad \text{for } j=1, \dots, m, \quad (1.15)$$

$$F_{m,0}(z) \equiv 1, \quad F_{m,1}(z) \equiv -(c_1(z))/m, \quad (1.16)$$

$$F_{m,j+1}(z) \equiv F'_{m,j}(z) + F_{m,1}(z)F_{m,j}(z), \quad \text{for } j=1, 2, 3, \dots, \quad (1.17)$$

$$G_{m,j}(z) \equiv \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} F_{m,j-i}(z), \quad \text{for } j=1, \dots, m \quad (1.18)$$

$$H_{m,j}(z) \equiv \sum_{i=3}^j B_{m,i}(z) \binom{m-i}{m-j} F_{m,j-i}(z), \quad \text{for } j=1, \dots, m, \quad (1.19)$$

$$I_{m,3}(z) \equiv H_{m,3}(z) \equiv c_3(z) - \xi_{m,3}(z), \quad \text{for } m \geq 3, \quad (1.20)$$

$$I_{m,4}(z) \equiv H_{m,4}(z) - \left(\frac{m-3}{2}\right) (I'_{m,3}(z)), \quad \text{for } m \geq 4, \quad (1.21)$$

$$\begin{aligned} I_{m,5}(z) &\equiv H_{m,5}(z) - \left(\frac{m-4}{2}\right) (I'_{m,4}(z)) - \left(\frac{(m-4)(m-3)}{7}\right) (I''_{m,3}(z)) \\ &\quad - \left(\frac{(m-4)(m-3)(7m+13)}{7(m+1)m(m-1)}\right) (G_{m,2}(z) I_{m,3}(z)), \quad \text{for } m \geq 5. \end{aligned} \quad (1.22)$$

To indicate how (1.20), (1.21), and (1.22) specify relative invariants, precise terminology is needed.

For any polynomial

$$P(w_1^{(0)}, w_1^{(1)}, \dots, w_m^{(0)}, w_m^{(1)}, \dots) \quad (1.23)$$

in the variables

$$w_j^{(k)}, \quad \text{for } j = 1, \dots, m \text{ and } k = 0, 1, 2, \dots, \quad (1.24)$$

over the field of complex numbers, let $I(z)$ denote the polynomial combination of

$$c_j^{(k)}(z), \quad \text{for } j = 1, \dots, m \text{ and } k = 0, 1, 2, \dots, \quad (1.25)$$

obtained when (1.25) is substituted for (1.24) in (1.23); let $I^*(z)$ denote the polynomial combination of

$$D_z^k c_j^*(z), \quad \text{for } j = 1, \dots, m \text{ and } k = 0, 1, 2, \dots, \quad (1.26)$$

obtained when (1.26) is substituted for (1.24) in (1.23); and let $I^{**}(\zeta)$ denote the polynomial combination of

$$D_\zeta^k c_j^{**}(\zeta), \quad \text{for } j = 1, \dots, m \text{ and } k = 0, 1, 2, \dots, \quad (1.27)$$

obtained when (1.27) is substituted for (1.24) in (1.23). Keeping m fixed and following standard usage, we say that a polynomial (1.23) is a *relative invariant* for differential equations of the form (1.1) when (1.23) is non-constant and there is an integer s such that the corresponding polynomial combinations $I(z)$ of (1.25) and $I^{**}(\zeta)$ of (1.27) satisfy

$$I^{**}(\zeta) \equiv (I(f(\zeta)))(f'(\zeta))^s, \quad (1.28)$$

for each Ω , each selection of $c_1(z), \dots, c_m(z)$ on Ω , each substitution (1.2) relative to Ω , and each ζ in V . An integer s that satisfies (1.28) is necessarily unique and, for reasons given later, is called the *weight* of (1.23).

When $c_1(z), \dots, c_m(z)$ are unrestricted and $I(z)$ is initially given as a polynomial combination of (1.25) having complex coefficients, one can immediately write down a unique polynomial (1.23) such that substitution of (1.25) for (1.24) in (1.23) yields $I(z)$. Then, $I^*(z)$ and $I^{**}(\zeta)$ are uniquely specified as before. Using this procedure, we shall establish in Proposition 2.14 that: for $m \geq 3$, (1.20) specifies a relative invariant of weight 3; for $m \geq 4$, (1.20) and (1.21) specify algebraically independent relative invariants of respective weights 3 and 4; and, for $m \geq 5$, (1.20), (1.21), (1.22) specify algebraically independent relative invariants of respective weights 3, 4, 5. Formulas (1.21) and (1.22) are a considerable improvement over previous efforts to specify relative invariants of weights 4 and 5 explicitly. And, our

methods can be continued. However, at this time, no one has discovered, for each $m \geq 3$, an algorithm to recursively define suitable expressions

$$I_{m,3}(z), I_{m,4}(z), \dots, I_{m,m}(z) \quad (1.29)$$

that explicitly specify $m - 2$ algebraically independent relative invariants of respective weights 3, 4, ..., m . This provides a very interesting subject for further research.

Of course, a polynomial (1.23) is a relative invariant if and only if it is a semi-invariant of both the first and second kinds according to the following definitions. A polynomial (1.23) is said to be a *semi-invariant of the first kind* when it is nonconstant and $I(z), I^*(z)$ satisfy $I^*(z) \equiv I(z)$, for each Ω , each selection of $c_1(z), \dots, c_m(z)$ on Ω , each substitution (1.2) relative to Ω having $f(\zeta) \equiv \zeta$, and each z in U . A polynomial (1.23) is said to be a *semi-invariant of the second kind* when it is nonconstant and there is an integer s such that $I(z), I^{**}(\zeta)$ satisfy (1.28) for each Ω , each selection of $c_1(z), \dots, c_m(z)$ on Ω , each substitution (1.2) relative to Ω having $\rho(z) \equiv 1$, and each ζ in V .

We shall establish in Theorem 2.13 the new result of basic importance that, for $m \geq 3$, the expressions $H_{m,3}(z), \dots, H_{m,m}(z)$ from (1.19) specify $m - 2$ algebraically independent semi-invariants of the first kind. As a considerably improved formulation of an old result, Proposition 2.12 shows that $G_{m,2}(z), \dots, G_{m,m}(z)$ from (1.18) specify semi-invariants of the first kind. Clearly, a nonconstant polynomial combination of semi-invariants of the first kind is a semi-invariant of the first kind. Writing $w_j^{(k+1)}$ for $(w_j^{(k)})'$, we see that the formal derivative of a semi-invariant of the first kind is a semi-invariant of the first kind. Thus, our plan for (1.29) when $m \geq 4$ is to start with $H_{m,3}(z), \dots, H_{m,m}(z)$ and to subtract from each of $H_{m,4}(z), \dots, H_{m,m}(z)$ suitable expressions specifying semi-invariants of the first kind so that the resulting $I_{m,3}(z), \dots, I_{m,m}(z)$ for (1.29) specify semi-invariants of both kinds. This is illustrated in (1.20), (1.21), (1.22).

G.-H. Halphen's contributions in [26, 30], for which he was awarded the 1880 *Grand Prix des Sciences Mathématiques* of the Academy of Sciences in Paris, strongly influenced later work on relative invariants. It is noteworthy that we have no need for any of the canonical forms of (1.1) or the corresponding restrictions on the substitutions (1.2) that G.-H. Halphen and others introduced when direct approaches appeared hopelessly difficult. However, the traditional assignment of weights is quite useful for our plan about (1.29) and will be summarized next.

Any nonconstant term of (1.23) can be uniquely written as

$$\gamma \prod_{v=1}^{\mu} w_v^{(k_v)}, \quad (1.30)$$

where γ is a nonzero complex number, μ is a positive integer, the j_v are integers satisfying

$$1 \leq j_1 \leq j_2 \leq \cdots \leq j_\mu \leq m,$$

and the k_v are nonnegative integers such that $k_v \leq k_{v+1}$ whenever $1 \leq v < \mu$ and $j_v = j_{v+1}$. Its *weight* g is defined to be the positive integer

$$g = \sum_{v=1}^{\mu} (j_v + k_v). \quad (1.31)$$

In agreement with (1.30) and (1.31) for $\mu = 0$, the weight of a nonzero constant is defined to be 0. We say that a polynomial (1.23) is *isobaric of weight* g when it is nonzero and can be expressed as a sum of nonzero terms each of weight g . Of course, a nonzero sum of two isobaric polynomials each of weight g is isobaric of weight g . And, the product of two isobaric polynomials of respective weights g_1 and g_2 is isobaric of weight $g_1 + g_2$. Writing $w_j^{(k+1)}$ for $(w_j^{(k)})'$, we see that the formal derivative of an isobaric polynomial of weight $g \geq 1$ is isobaric of weight $g + 1$.

The isobaric polynomial $w_j^{(k)}$ has weight $g = j + k$ and it specifies $c_j^{(k)}(z)$, $D_z^k c_j^*(z)$, and $D_\zeta^k c_j^{**}(\zeta)$ through (1.25), (1.26), and (1.27). For unrestricted $c_1(z)$, ..., $c_m(z)$ and any one of the polynomial combinations

$$\begin{aligned} &\alpha(z), \beta(z), a_{i,j}(z), \xi_{m,j}(z), B_{m,j}(z), \\ &F_{m,j}(z), G_{m,j}(z), H_{m,j}(z), I_{m,j}(z) \end{aligned} \quad (1.32)$$

of (1.25) in (1.11) through (1.22), there is a unique polynomial (1.23) from which it is obtained by substitution of (1.25) for (1.24) in (1.23). And, when nonzero, it is isobaric of weight g , where: for $\alpha(z)$, $g = 1$; for $\beta(z)$, $g = 2$; for $a_{i,j}(z)$ with $1 \leq i$ and $0 \leq j \leq i + 1$, $g = i - j + 1$; for $\xi_{m,j}(z)$, $g = j$; for $B_{m,j}(z)$ and $3 \leq j \leq m$, $g = j$; for $F_{m,j}(z)$, $g = j$; for $G_{m,j}(z)$ and $2 \leq j \leq m$, $g = j$; for $H_{m,j}(z)$ and $3 \leq j \leq m$, $g = j$; for $I_{m,j}(z)$, $g = j$.

Simple arguments in [26] or [30] show that: any semi-invariant of the second kind having s as an integer for (1.28) is an isobaric polynomial whose weight g satisfies $g = s \geq 2$; and any relative invariant having s as an integer for (1.28) is an isobaric polynomial whose weight g satisfies $g = s \geq 3$. Thus, in our plan to obtain $I_{m,j}(z)$ for (1.29) when $4 \leq j \leq m$, the expressions to be subtracted from $H_{m,j}(z)$ must specify an isobaric semi-invariant of the first kind having weight j . This is illustrated for $j = 4$ and $j = 5$ in (1.21) and (1.22).

For later reference, we let

$$\begin{aligned} &\alpha^*(z), \beta^*(z), a_{i,j}^*(z), \xi_{m,j}^*(z), B_{m,j}^*(z), \\ &F_{m,j}^*(z), G_{m,j}^*(z), H_{m,j}^*(z), I_{m,j}^*(z) \end{aligned} \quad (1.33)$$

denote the expressions obtained when (1.26) is substituted for (1.24) in the various polynomials that specify (1.32). Similarly, we let

$$\begin{aligned} \alpha^{**}(\zeta), \beta^{**}(\zeta), a_{i,j}^{**}(\zeta), \xi_{m,j}^{**}(\zeta), B_{m,j}^{**}(\zeta), \\ F_{m,j}^{**}(\zeta), G_{m,j}^{**}(\zeta), H_{m,j}^{**}(\zeta), I_{m,j}^{**}(\zeta) \end{aligned} \quad (1.34)$$

denote the expressions obtained when (1.27) is substituted for (1.24) in the same polynomials.

Our immediate goals for Section 2 are to relate the $B_{m,j}^*(z)$ to the $B_{m,j}(z)$ by means of (1.2) and then to relate the $B_{m,j}^{**}(\zeta)$ to the $B_{m,j}^*(z)$ by means of (1.2). As we shall see, this requires new results from Section 4. However, Sections 3, 4, and 5 are independent of Sections 1 and 2.

2. BASIC RESULTS ABOUT RELATIVE INVARIANTS

For a fixed integer $m \geq 2$, we continue the context of Section 1 and introduce the $m \times m$ matrix

$$M_m(z) \equiv \begin{bmatrix} -c_1(z) & -c_2(z) & \cdots & -c_{m-1}(z) & -c_m(z) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (2.1)$$

so that (1.1) can be replaced with

$$[y'_{m-1}, y'_{m-2}, \dots, y'_0]^T = (M_m(z))[y_{m-1}, y_{m-2}, \dots, y_0]^T, \quad (2.2)$$

where $y_i = y^{(i)}$, for $i=0, 1, \dots, m-1$, and T denotes matrix transposition. Substituting $c_j^*(z)$ for $c_j(z)$ in the right member of (2.1), we obtain an $m \times m$ matrix $M_m^*(z)$ that enables us to replace (1.5) with

$$[v'_{m-1}, v'_{m-2}, \dots, v'_0]^T = (M_m^*(z))[v_{m-1}, v_{m-2}, \dots, v_0]^T, \quad (2.3)$$

where $v_i = v^{(i)}$, for $i=0, 1, \dots, m-1$. Substituting $c_j^{**}(\zeta)$ for $c_j(z)$ in the right member of (2.1), we obtain an $m \times m$ matrix $M_m^{**}(\zeta)$ that enables us to replace (1.6) with

$$[t'_{m-1}, t'_{m-2}, \dots, t'_0]^T = (M_m^{**}(\zeta))[t_{m-1}, t_{m-2}, \dots, t_0]^T, \quad (2.4)$$

where $t_i = D_\zeta^i v(f(\zeta))$, for $i=0, 1, \dots, m-1$.

PROPOSITION 2.1. *The coefficients $c_1^*(z), \dots, c_m^*(z)$ of (1.5) are given on U by*

$$c_j^*(z) \equiv \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} \frac{\rho^{(j-i)}(z)}{\rho(z)}, \quad \text{for } j = 1, \dots, m. \quad (2.5)$$

Proof. Let $L(z, y, y', \dots, y^{(m)})$ and $L^*(z, v, v', \dots, v^{(m)})$ denote the left members of (1.1) and (1.5). Using (1.7), we find that

$$\begin{aligned} & L(z, \rho(z)v, (\rho(z)v)', \dots, (\rho(z)v)^{(m)}) \\ & \equiv \sum_{i=0}^m c_i(z) \sum_{k=0}^{m-1} \binom{m-i}{k} (\rho^{(m-i-k)}(z))(v^{(k)}) \\ & \equiv \sum_{k=0}^m \sum_{i=0}^{m-k} c_i(z) \binom{m-i}{k} (\rho^{(m-i-k)}(z))(v^{(k)}) \\ & \equiv \sum_{j=0}^m \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} (\rho^{(j-i)}(z))(v^{(m-j)}) \\ & \equiv \rho(z) L^*(z, v, v', \dots, v^{(m)}), \end{aligned}$$

where $c_j^*(z)$ in L^* is given by (2.5). This completes the proof.

The transformation (1.3) uniquely specifies $m \times m$ matrices

$$P_m(z) \equiv [p_{i,j}(z)] \quad \text{and} \quad Q_m(z) \equiv [q_{i,j}(z)] \quad (2.6)$$

whose components, for $i, j = 1, 2, \dots, m$, are the analytic functions on U given by

$$\begin{aligned} p_{i,j}(z) & \equiv \binom{m-i}{m-j} \frac{\rho^{(j-i)}(z)}{\rho(z)}, & \text{when } i \leq j, \\ q_{i,j}(z) & \equiv -\binom{m-i+1}{m-j} \frac{\rho^{(j-i+1)}(z)}{\rho(z)}, & \text{when } i \leq j, \end{aligned}$$

and $p_{i,j}(z) \equiv q_{i,j}(z) \equiv 0$, when $i > j$. In particular, $P_m(z)$ is a nonsingular upper triangular matrix and Q_m is upper triangular.

PROPOSITION 2.2. *The companion matrices $M_m(z)$ and $M_m^*(z)$ in (2.2) and (2.3) are related on U by*

$$M_m^*(z) \equiv (M_m(z))(P_m(z)) + Q_m(z). \quad (2.7)$$

Proof. For $j = 1, \dots, m$, we use (2.5) to obtain

$$-c_j^*(z) \equiv \left(\sum_{i=1}^m (-c_i(z)) p_{i,j}(z) \right) + q_{1,j}(z).$$

Thus, the $(1, j)$ components of the two members of (2.7) are equal. For $i \geq 2$, the (i, j) component of the right member of (2.7) is

$$\lambda_{i,j}(z) \equiv p_{i-1,j}(z) + q_{i,j}(z).$$

If $i < j + 1$, then $p_{i-1,j} \equiv -q_{i,j}$ and $\lambda_{i,j} \equiv 0$. If $i = j + 1$, then $p_{i-1,j} \equiv 1$, $q_{i,j} \equiv 0$, and $\lambda_{i,j} \equiv 1$. If $i > j$, then $p_{i-1,j} \equiv 0$, $q_{i,j} \equiv 0$, and $\lambda_{i,j} \equiv 0$. This yields (2.7) and completes the proof.

Recalling (1.7), we define $\xi_{m,0}(z)$ on Ω , $\xi_{m,0}^*(z)$ on U , and $\xi_{m,0}^{**}(\zeta)$ on V by

$$\xi_{m,0}(z) \equiv 1, \quad \xi_{m,0}^*(z) \equiv 1, \quad \xi_{m,0}^{**}(\zeta) \equiv 1, \quad (2.8)$$

in order to simplify later formulas such as (2.11). And, we let E_Ω , E_U , E_V denote the fields of meromorphic functions defined on Ω , U , V , respectively.

THEOREM 2.3. *The expressions $\alpha^*(z)$, $\beta^*(z)$, $\xi_{m,j}^*(z)$ in (1.33) for $m \geq 2$ are given on U by*

$$\alpha^*(z) \equiv \alpha(z) + \frac{2\rho'(z)}{(m-1)\rho(z)}, \quad (2.9)$$

$$\beta^*(z) \equiv \beta(z) + \frac{\rho'(z)\alpha(z)}{(m-1)\rho(z)} + \frac{(2-m)(\rho'(z))^2}{(m-1)^2(\rho(z))^2} + \frac{\rho''(z)}{(m-1)\rho(z)}, \quad (2.10)$$

$$\xi_{m,j}^*(z) \equiv \sum_{i=0}^j \xi_{m,i}(z) \binom{m-i}{m-j} \frac{\rho^{(j-i)}(z)}{\rho(z)}, \quad \text{for } j = 1, \dots, m. \quad (2.11)$$

Proof. Substituting $j = 1$ and $j = 2$ in (2.5), we find

$$c_1^*(z) \equiv c_1(z) + \frac{m\rho'(z)}{\rho(z)} \quad (2.12)$$

and

$$c_2^*(z) \equiv c_2(z) + c_1(z) \frac{(m-1)\rho'(z)}{\rho(z)} + \binom{m}{2} \frac{\rho''(z)}{\rho(z)}. \quad (2.13)$$

We use the formula $\alpha^*(z) \equiv (2c_1^*(z))/(m(m-1))$, corresponding to (1.12), as well as (2.12) to obtain (2.9). The formula

$$\beta^*(z) \equiv \frac{\left(6c_2^*(z) - 2(m-2)(D_z c_1^*(z)) - ((3m-1)(m-2)/m(m-1))(c_1^*(z))^2 \right)}{(m+1)m(m-1)} \quad (2.14)$$

corresponds to (1.13). Combining (2.14), (2.13), and (2.12), we express $\beta^*(z)$ in terms of $c_2(z)$ and $c_1(z)$ on U . Then, we involve (1.13) and (1.12) on U to eliminate $c_2(z)$, $c_1(z)$ and obtain (2.10).

We set $n = m - 1$ and introduce

$$Q_n^*(z, v, v', v'') \equiv v''v + \left(\frac{1-n}{n}\right)v'^2 + \alpha^*(z)v'v + n\beta^*(z)v^2 \quad (2.15)$$

in terms of $\alpha^*(z)$ and $\beta^*(z)$ on U . When the substitution

$$y = \rho(z)v, \quad y' = \rho(z)v' + \rho'(z)v, \dots \quad (2.16)$$

is applied to the restriction to U of

$$Q_n(z, y, y', y'') \equiv y''y + \left(\frac{1-n}{n}\right)y'^2 + \alpha(z)y'y + n\beta(z)y^2,$$

formulas (2.9), (2.10), (2.15) yield

$$Q_n(z, y, y', y'') \equiv (\rho(z))^2 Q_n^*(z, v, v', v'') \quad (2.17)$$

over E_U . Now, we consider the identity (4.3) in Theorem 4.1 for our present Q_n and the context where E is E_Ω and $n = m - 1$. In particular, (4.4) and (4.20) for (4.19) reduce to (1.14) and (1.11) for (1.9). Restricting the coefficients of our particular (4.3) to U , interpreting this identity as one over E_U , and multiplying both of its members by $(1/\rho(z))^{n+1}$, we apply (2.16), (2.17) to obtain

$$\sum_{j=0}^{n-1} (P_{n,j}^*)(D_z^j Q_n^*) \equiv v^n L_n^*(z, v, v', \dots, v^{(n+1)}), \quad (2.18)$$

where the $P_{n,j}^*$ are polynomials in v, v', \dots over E_U and

$$\begin{aligned} L_n^* &\equiv (1/\rho(z)) \sum_{j=0}^{n+1} a_{n,j}(z)(\rho(z)v)^{(j)} \\ &\equiv (1/\rho(z)) \sum_{i=0}^m \xi_{m,i}(z)(D_z^{m-i}(\rho(z)v)) \\ &\equiv (1/\rho(z)) \sum_{i=0}^m \xi_{m,i}(z) \sum_{k=0}^{m-i} \binom{m-i}{k} (D_z^{m-i-k}\rho(z))v^{(k)} \\ &\equiv (1/\rho(z)) \sum_{k=0}^m \sum_{i=0}^{m-k} \xi_{m,i}(z) \binom{m-i}{k} (D_z^{m-i-k}\rho(z))v^{(k)} \\ &\equiv \sum_{j=0}^m \left(\sum_{i=0}^j \xi_{m,i}(z) \binom{m-i}{m-j} \frac{\rho^{(j-i)}(z)}{\rho(z)} \right) v^{(m-j)}. \end{aligned} \quad (2.19)$$

However, an application of Theorem 4.2 to the identity (2.18) for Q_n^* in (2.15) and $n = m - 1$ shows that

$$L_n^* \equiv \sum_{j=0}^m \xi_{m,j}^*(z) v^{(m-j)}. \quad (2.20)$$

We equate the coefficients of $v^{(m-j)}$ in (2.20) and (2.19) to obtain (2.11). This completes the proof.

COROLLARY 2.4. *The $1 \times m$ row vectors*

$$J_m(z) \equiv [B_{m,1}(z), \dots, B_{m,m}(z)] \quad (2.21)$$

and

$$J_m^*(z) \equiv [B_{m,1}^*(z), \dots, B_{m,m}^*(z)] \quad (2.22)$$

are related on U through

$$J_m^*(z) \equiv (J_m(z))(P_m(z)) \quad (2.23)$$

by means of the $m \times m$ matrix $P_m(z)$ in (2.6).

Proof. Subtracting (2.11) from (2.5), we use (1.7), (2.8), and the components $p_{i,j}(z)$ of $P_m(z)$ to obtain

$$c_j^*(z) - \xi_{m,j}^*(z) \equiv \sum_{i=1}^m (c_i(z) - \xi_{m,i}(z))(p_{i,j}(z)),$$

for $j = 1, \dots, m$. This yields (2.23) and completes the proof.

COROLLARY 2.5. *For each z in Ω ,*

$$B_{m,1}(z) \equiv 0, \quad B_{m,2}(z) \equiv 0, \quad (2.24)$$

and, when $m \geq 3$,

$$\begin{aligned} B_{m,3}(z) \equiv & c_3(z) + \left(\frac{(m-1)(m-2)}{3m^2} \right) (c_1(z))^3 + \left(\frac{(m-1)(m-2)}{2m} \right) c_1(z) c_1'(z) \\ & + \left(\frac{(m-1)(m-2)}{12} \right) c_1''(z) - \left(\frac{m-2}{m} \right) c_1(z) c_2(z) \\ & - \left(\frac{m-2}{2} \right) c_2'(z). \end{aligned} \quad (2.25)$$

Moreover, for each z in U , $B_{m,1}^*(z) \equiv B_{m,2}^*(z) \equiv 0$ and

$$B_{m,3}^*(z) \equiv B_{m,3}(z), \quad \text{when } m \geq 3. \quad (2.26)$$

Proof. Proposition 4.4 yields (2.24). The definitions for (1.33) then give $B_{m,1}^*(z) \equiv B_{m,2}^*(z) \equiv 0$. Suppose $m \geq 3$. Setting $n = m - 1$ in (4.28), we use (1.12) and (1.13) to write $a_{m-1, m-3}(z)$ as a polynomial combination of $c_1(z)$, $c'_1(z)$, $c''_1(z)$, $c_2(z)$, $c'_2(z)$. This expression for $\xi_{m,3}(z)$ is subtracted from $c_3(z)$ to obtain (2.25). We equate the (1, 3) components in (2.23) to deduce (2.26) and complete the proof.

The univalent analytic function f on V in (1.2) maps V onto U . We necessarily have $f'(z) \neq 0$, for each ζ in V . Let g be the inverse function for f . It follows that g is a univalent analytic function on U mapping U onto V and satisfying: $f(g(z)) = z$, for each z in U ; $g(f(\zeta)) = \zeta$, for each ζ in V ; $g'(z) \neq 0$, for each z in U ; and

$$g'(f(\zeta)) \equiv 1/(f'(\zeta)), \quad \text{for each } \zeta \text{ in } V. \quad (2.27)$$

For later reference, we also have

$$g''(f(\zeta)) \equiv \frac{-f''(\zeta)}{(f'(\zeta))^3} \quad (2.28)$$

and

$$g'''(f(\zeta)) \equiv \frac{-f'''(\zeta)}{(f'(\zeta))^4} + \frac{3(f''(\zeta))^2}{(f'(\zeta))^5}, \quad (2.29)$$

for each ζ in V .

For each $v(z)$ in E_U and each ζ in V , we have

$$v'(f(\zeta)) \equiv (g'(f(\zeta)))(D_\zeta(v \circ f)(\zeta)) \quad (2.30)$$

and

$$v''(f(\zeta)) \equiv (g'(f(\zeta)))^2 (D_\zeta^2(v \circ f)(\zeta)) + (g''(f(\zeta)))(D_\zeta(v \circ f)(\zeta)). \quad (2.31)$$

Analytic functions $u_{i,j}(z)$ on U , for $i, j = 0, 1, 2, \dots$, are defined in E_U by

$$\begin{aligned} u_{0,0}(z) &\equiv 1; \\ u_{i,0}(z) &\equiv 0, \quad \text{for } i = 1, 2, 3, \dots; \\ u_{i,j}(z) &\equiv 0, \quad \text{for } i, j = 0, 1, 2, \dots \text{ and } i < j; \\ u_{i,j}(z) &\equiv u'_{i-1,j}(z) + (g'(z))(u_{i-1,j-1}(z)), \\ &\quad \text{for } i, j = 1, 2, 3, \dots \text{ and } i \geq j. \end{aligned} \quad (2.32)$$

Using (2.32) and writing $(v^{(i)}(z))_{z=f(\zeta)}$ for $v^{(i)}(f(\zeta))$, we find that

$$(v^{(i)}(z))_{z=f(\zeta)} \equiv \sum_{j=0}^i (u_{i,j}(f(\zeta)))(D_{\zeta}^j(v \circ f)(\zeta)),$$

for $i = 0, 1, 2, \dots$ and each ζ in V . (2.33)

Applying the calculus of finite differences to (2.32) on U , we obtain

$$u_{i,i}(z) \equiv (g'(z))^i, \quad \text{for } i = 0, 1, 2, \dots, \quad (2.34)$$

$$u_{i,i-1}(z) \equiv \binom{i}{2} (g'(z))^{i-2} (g''(z)), \quad \text{for } i = 1, 2, 3, \dots, \quad (2.35)$$

$$u_{i,i-2}(z) \equiv \binom{i}{3} (g'(z))^{i-3} (g'''(z)) + 3 \binom{i}{4} (g'(z))^{i-4} (g''(z))^2,$$

for $i = 2, 3, 4, \dots$ (2.36)

PROPOSITION 2.6. *The coefficients $c_1^{**}(\zeta), \dots, c_m^{**}(\zeta)$ of (1.6) are given on V by*

$$c_j^{**}(\zeta) \equiv \sum_{i=0}^j (c_i^*(f(\zeta)))(f'(\zeta))^m (u_{m-i, m-j}(f(\zeta))), \quad \text{for } j = 1, \dots, m. \quad (2.37)$$

Proof. Let $L^*(z, v, v', \dots, v^{(m)})$ and $L^{**}(\zeta, t, t', \dots, t^{(m)})$ denote the left members of (1.5) and (1.6). Using (1.5), (1.7), (2.33), (2.34), (2.27), and (1.6), we find that

$$\begin{aligned} & (L^*(z, v(z), v'(z), \dots, v^{(m)}(z)))_{z=f(\zeta)} \\ & \equiv \sum_{i=0}^m (c_i^*(f(\zeta))) \sum_{k=0}^{m-i} (u_{m-i, k}(f(\zeta)))(D_{\zeta}^k(v \circ f)(\zeta)) \\ & \equiv \sum_{k=0}^m \sum_{i=0}^{m-k} (c_i^*(f(\zeta)))(u_{m-i, k}(f(\zeta)))(D_{\zeta}^k(v \circ f)(\zeta)) \\ & \equiv \sum_{j=0}^m \sum_{i=0}^j (c_i^*(f(\zeta)))(u_{m-i, m-j}(f(\zeta)))(D_{\zeta}^{m-j}(v \circ f)(\zeta)) \\ & \equiv (g'(f(\zeta)))^m (L^{**}(\zeta, t(\zeta), t'(\zeta), \dots, t^{(m)}(\zeta))), \end{aligned}$$

where $t(\zeta) = (v \circ f)(\zeta)$. Thus, $c_j^{**}(\zeta)$ is given by (2.37). This completes the proof.

The transformation (1.4) uniquely specifies $m \times m$ matrices

$$R_m(\zeta) \equiv [r_{i,j}(\zeta)] \quad \text{and} \quad S_m(\zeta) \equiv [s_{i,j}(\zeta)] \quad (2.38)$$

whose components, for $i, j = 1, 2, \dots, m$, are the analytic functions on V given by

$$r_{i,j}(\zeta) \equiv (f'(\zeta))^m (u_{m-i, m-j}(f(\zeta)))$$

and

$$s_{i,j}(\zeta) \equiv \delta_{i-1,j} - (f'(\zeta))^m (u_{m+1-i, m-j}(f(\zeta))).$$

Here, for any integers μ and ν , $\delta_{\mu,\nu}$ equals 1 when $\mu = \nu$ and it equals 0 when $\mu \neq \nu$. Due to (2.32) and (2.34), $R_m(\zeta)$ is a *nonsingular upper triangular matrix*. But, $S_m(\zeta)$ is upper triangular if and only if $f'(\zeta) \equiv 1$ and $f(\zeta) \equiv \zeta + C$, for some constant C .

PROPOSITION 2.7. *The companion matrices $M_m^*(z)$ and $M_m^{**}(\zeta)$ in (2.3) and (2.4) are related on V by*

$$M_m^{**}(\zeta) \equiv (M_m^*(f(\zeta)))(R_m(\zeta)) + S_m(\zeta). \quad (2.39)$$

Proof. For $j = 1, \dots, m$, we use (2.37) and (2.32) to obtain

$$-c_j^{**}(\zeta) \equiv \left(\sum_{i=1}^m (-c_i^*(f(\zeta)))(r_{i,j}(\zeta)) \right) + s_{1,j}(\zeta).$$

Thus, the $(1, j)$ components of the two members of (2.39) are equal. For $i \geq 2$, the (i, j) component of the right member of (2.39) equals

$$\left(\sum_{k=1}^m \delta_{i-1,k} r_{k,j}(\zeta) \right) + s_{i,j}(\zeta) \equiv r_{i-1,j}(\zeta) + s_{i,j}(\zeta) \equiv \delta_{i-1,j}.$$

This yields (2.39) and completes the proof.

THEOREM 2.8. *The expressions $\alpha^{**}(\zeta)$, $\beta^{**}(\zeta)$, $\xi_{m,j}^{**}(\zeta)$ in (1.34) for $m \geq 2$ are given on V by*

$$\alpha^{**}(\zeta) \equiv (\alpha^*(f(\zeta)))(f'(\zeta)) - \frac{f''(\zeta)}{f'(\zeta)}, \quad (2.40)$$

$$\beta^{**}(\zeta) \equiv (\beta^*(f(\zeta)))(f'(\zeta))^2, \quad (2.41)$$

$$\xi_{m,j}^{**}(\zeta) \equiv \sum_{i=0}^j (\xi_{m,i}^*(f(\zeta)))(f'(\zeta))^m (u_{m-i, m-j}(f(\zeta))),$$

for $j = 1, \dots, m$. (2.42)

Proof. Using (2.37) for $j = 1$, (2.34), (2.35), (2.27), and (2.28), we obtain

$$c_1^{**}(\zeta) \equiv (c_1^*(f(\zeta)))(f'(\zeta)) - \left(\frac{m}{2} \right) \frac{f''(\zeta)}{f'(\zeta)}, \quad (2.43)$$

for each ζ in V . Application of (2.37) for $j=2$, (2.34), (2.35), (2.36), (2.27), (2.28), and (2.29) shows that, for each ζ in V ,

$$\begin{aligned} c_2^{**}(\zeta) &\equiv (c_2^*(f(\zeta)))(f'(\zeta))^2 - (c_1^*(f(\zeta))) \binom{m-1}{2} (f''(\zeta)) \\ &\quad - \binom{m}{3} \left(\frac{f'''(\zeta)}{f'(\zeta)} \right) + 3 \binom{m+1}{4} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2. \end{aligned} \quad (2.44)$$

The formulas $\alpha^*(z) \equiv (2c_1^*(z))/(m(m-1))$ and $\alpha^{**}(\zeta) \equiv (2c_1^{**}(\zeta))/(m(m-1))$ correspond to (1.12). We use them and (2.43) to obtain (2.40). The formulas (2.14) and

$$\beta^{**}(\zeta) \equiv \frac{\left(6c_2^{**}(\zeta) - 2(m-2)(D_\zeta c_1^{**}(\zeta)) - ((3m-1)(m-2)/m(m-1))(c_1^{**}(\zeta))^2 \right)}{(m+1)m(m-1)}, \quad (2.45)$$

for $\beta^*(z)$ on U and $\beta^{**}(\zeta)$ on V , correspond to (1.13). We use (2.45), (2.44), (2.43), and (2.14) to deduce (2.41).

Setting $n = m - 1$, we apply (2.30), (2.31), (2.40), (2.41) to Q_n^* in (2.15) and verify that, for each ζ in V ,

$$\begin{aligned} (Q_n^*(z, v(z), v'(z), v''(z)))_{z=f(\zeta)} \\ \equiv (g'(f(\zeta)))^2 (Q_n^{**}(\zeta, t(\zeta), t'(\zeta), t''(\zeta))), \end{aligned} \quad (2.46)$$

where $t(\zeta) = v(f(\zeta)) = (v \circ f)(\zeta)$ and

$$Q_n^{**}(\zeta, t, t', t'') \equiv t''t + \left(\frac{1-n}{n} \right) t'^2 + \alpha^{**}(\zeta) t't + n\beta^{**}(\zeta) t^2. \quad (2.47)$$

The identity (2.18) for Q_n^* in (2.15) and L_n^* in (2.20) yields

$$\begin{aligned} (f'(\zeta))^m \left(\sum_{j=1}^{n-1} (P_{n,j}^*)(D_z^j Q_n^*) \right)_{z=f(\zeta)} \\ \equiv (f'(\zeta))^m (v(f(\zeta)))^n (L_n^*)_{z=f(\zeta)}, \end{aligned} \quad (2.48)$$

for each ζ in V . Due to (2.46) and (2.33), we see that

$$\begin{aligned} (D_z^j Q_n^*(z, v(z), v'(z), v''(z)))_{z=f(\zeta)} \\ \equiv \sum_{k=0}^j (u_{j,k}(f(\zeta))) \{ D_\zeta^k [(g'(f(\zeta)))^2 (Q_n^{**}(\zeta, t(\zeta), t'(\zeta), t''(\zeta)))] \}, \end{aligned}$$

for $j=0, 1, 2, \dots$ and each ζ in V . Thus, polynomial combinations $P_{n,j}^{**}$ of $t(\zeta), t'(\zeta), \dots$ over E_V exist such that (2.48) can be rewritten on V as

$$\sum_{j=0}^{n-1} (P_{n,j}^{**})(D_{\zeta}^j Q_n^{**}(\zeta, t(\zeta), t'(\zeta), t''(\zeta))) \equiv (t(\zeta))^n (L_n^{**}), \quad (2.49)$$

where

$$\begin{aligned} L_n^{**} &\equiv (f'(\zeta))^m (L_n^*)_z = f(\zeta) \\ &\equiv (f'(\zeta))^m \sum_{i=0}^m (\xi_{m,i}^{**}(f(\zeta))) \sum_{k=0}^{m-i} (u_{m-i,k}(f(\zeta)))(D_{\zeta}^k(v \circ f)(\zeta)) \\ &\equiv (f'(\zeta))^m \sum_{k=0}^m \sum_{i=0}^{m-k} (\xi_{m,i}^{**}(f(\zeta)))(u_{m-i,k}(f(\zeta)))(D_{\zeta}^k(v \circ f)(\zeta)) \\ &\equiv D_{\zeta}^m t(\zeta) + \sum_{j=1}^m (\eta_{m,j}^{**}(\zeta))(D_{\zeta}^{m-j} t(\zeta)) \end{aligned} \quad (2.50)$$

and $\eta_{m,j}^{**}(\zeta)$ is given by the right member of (2.42). However, an application of Theorem 4.2 to the identity (2.49) for Q_n^{**} in (2.47) and the context where E is E_V , $'$ is D_{ζ} , and $n=m-1$ shows that

$$L_n^{**} \equiv D_{\zeta}^m t(\zeta) + \sum_{j=1}^m (\xi_{m,j}^{**}(\zeta))(D_{\zeta}^{m-j} t(\zeta)), \quad (2.51)$$

for each ζ in V . We equate the coefficients of $(D_{\zeta}^{m-j} t(\zeta))$ in (2.51) and (2.50) to obtain (2.42). This completes the proof.

COROLLARY 2.9. *The $1 \times m$ row vectors*

$$J_m^{**}(\zeta) \equiv [B_{m,1}^{**}(\zeta), \dots, B_{m,m}^{**}(\zeta)] \quad (2.52)$$

and $J_m^*(z)$ in (2.22) are related on V through

$$J_m^{**}(\zeta) \equiv (J_m^*(f(\zeta)))(R_m(\zeta)) \quad (2.53)$$

by means of the $m \times m$ matrix $R_m(\zeta)$ in (2.38).

Proof. Subtracting (2.42) from (2.37), we use (1.7), (2.8), and the components $r_{i,j}(\zeta)$ of $R_m(\zeta)$ to obtain

$$c_j^{**}(\zeta) - \xi_{m,j}^{**}(\zeta) \equiv \sum_{i=1}^m (c_i^*(f(\zeta)) - \xi_{m,i}^*(f(\zeta)))(r_{i,j}(\zeta)),$$

for $j=1, \dots, m$. This yields (2.53) and completes the proof.

COROLLARY 2.10. For each ζ in V ,

$$B_{m,3}^{**}(\zeta) \equiv (B_{m,3}(f(\zeta)))(f'(\zeta))^3, \quad \text{when } m \geq 3. \quad (2.54)$$

Proof. Using $B_{m,1}^*(z) \equiv B_{m,2}^*(z) \equiv 0$ and (2.26) of Corollary 2.5, we equate the (1, 3) components of (2.53) to obtain (2.54). This completes the proof.

Formula (2.54) shows that a relative invariant of weight 3 is specified by $B_{m,3}(z)$ when $m \geq 3$. Dividing $B_{m,3}(z)$ in (2.25) by $m^3 - m$, we obtain an expression analogous to one in [26, p. 127, Eq. (10)] or [30, p. 112, Eq. (10)] for $m \geq 3$ and one in [32, p. 117] or [27, p. 421 (misprinted)] for $m = 3$.

To derive further relative invariants when $m \geq 4$, we include here for later reference the formulas

$$B_{m,j}^*(z) \equiv \sum_{i=3}^j B_{m,i}(z) \binom{m-i}{m-j} \frac{\rho^{(j-i)}(z)}{\rho(z)}, \quad \text{for } j = 1, \dots, m, \quad (2.55)$$

and

$$B_{m,j}^{**}(\zeta) \equiv \sum_{i=3}^j (B_{m,i}^*(f(\zeta)))(f'(\zeta))^m (u_{m-i, m-j}(f(\zeta))),$$

for $j = 1, \dots, m,$ (2.56)

obtained from (2.23) and (2.53) for $m \geq 2$. The following result will be used in the proofs of Proposition 2.12 and Theorem 2.13.

LEMMA 2.11. Given $\rho(z)$ in (1.2), formulas (1.16), (1.17), and (1.33) yield

$$\sum_{v=0}^{\mu} \binom{\mu}{v} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m, \mu-v}^* \equiv F_{m, \mu}(z), \quad (2.57)$$

on U for $\mu = 0, 1, 2, \dots$.

Proof. For $\mu = 0, 1, 2, \dots$, let $S_{\mu}(z)$ denote the left member of (2.57). Using (1.16), (1.33), and (2.12), we deduce $S_0(z) \equiv 1 \equiv F_{m,0}(z)$ and

$$S_1(z) \equiv (-c_1^*(z)/m) + \frac{\rho'(z)}{\rho(z)} \equiv (-c_1(z)/m) \equiv F_{m,1}(z).$$

Suppose $S_{\mu}(z) \equiv F_{m,\mu}(z)$, for some $\mu \geq 1$. Using this with (1.16), (1.17), (1.33), and (2.12), we obtain

$$\begin{aligned}
F_{m, \mu+1}(z) &\equiv F'_{m, \mu}(z) + F_{m, 1}(z) F_{m, \mu}(z) \\
&\equiv S'_\mu(z) + \left(\frac{\rho'(z)}{\rho(z)} + F_{m, 1}^*(z) \right) S_\mu(z) \\
&\equiv \sum_{v=0}^{\mu} \binom{\mu}{v} \frac{\rho^{(v)}(z)}{\rho(z)} (D_z F_{m, \mu-v}^*(z) + F_{m, 1}^*(z) F_{m, \mu-v}^*(z)) \\
&\quad + \sum_{v=0}^{\mu} \binom{\mu}{v} \frac{\rho^{(v+1)}(z)}{\rho(z)} F_{m, \mu-v}^*(z) \\
&= \sum_{v=0}^{\mu} \binom{\mu}{v} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m, \mu+1-v}^*(z) + \sum_{v=1}^{\mu+1} \binom{\mu}{v-1} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m, \mu+1-v}^*(z) \\
&\equiv \sum_{v=0}^{\mu+1} \binom{\mu+1}{v} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m, \mu+1-v}^*(z) \\
&\equiv S_{\mu+1}(z).
\end{aligned}$$

Thus, we have $S_\mu(z) \equiv F_{m, \mu}(z)$, for all nonnegative integers μ . This completes the proof.

The following precise formulation of a classical result will be needed for Proposition 2.14.

PROPOSITION 2.12. *On U , (1.18) and (1.33) yield $G_{m, 1}(z) \equiv 0$ and*

$$G_{m, j}^*(z) \equiv G_{m, j}(z), \quad \text{for } j = 1, 2, \dots, m. \quad (2.58)$$

Proof. To deduce $G_{m, 1}(z) \equiv 0$, we combine (1.18) for $j=1$ with (1.7) and (1.16). We use (1.18), the analogous formula for $G_{m, j}^*(z)$ in (1.33), (2.5), (1.7), and (2.57) to obtain

$$\begin{aligned}
G_{m, j}^*(z) &\equiv \sum_{k=0}^j c_k^*(z) \binom{m-k}{m-j} F_{m, j-k}^*(z) \\
&\equiv \sum_{k=0}^j \sum_{i=0}^k c_i(z) \binom{m-i}{m-k} \frac{\rho^{(k-i)}(z)}{\rho(z)} \binom{m-k}{m-j} F_{m, j-k}^*(z) \\
&\equiv \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} \sum_{k=i}^j \binom{j-i}{j-k} \frac{\rho^{(k-i)}(z)}{\rho(z)} F_{m, j-k}^*(z) \\
&\equiv \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} \sum_{v=0}^{j-i} \binom{j-i}{v} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m, j-i-v}^*(z) \\
&\equiv \sum_{i=0}^j c_i(z) \binom{m-i}{m-j} F_{m, j-i}(z) \\
&\equiv G_{m, j}(z),
\end{aligned}$$

on U for $j = 1, 2, \dots, m$. This completes the proof.

Our formula (1.18) for the $G_{m,j}(z)$ is new and much more explicit than earlier definitions; e.g., compare [43, pp. 287–288; 7, p. 30; 49, pp. 14–17]. In fact, it is not obvious what $G_{m,j}(z)$ corresponds to in previous publications. To explain this and to clarify the standard argument for (2.58), we include the following proof.

Alternate Proof of Proposition 2.12. There are analytic functions $\sigma(z)$, $\tau(z)$ defined and free of zeros on a subregion U_0 of U such that

$$\frac{\sigma'(z)}{\sigma(z)} \equiv -(c_1(z))/m \quad \text{and} \quad \frac{\tau'(z)}{\tau(z)} \equiv -(c_1^*(z))/m,$$

for each z in U_0 . Formulas (1.16), (1.17), and analogous ones for (1.33) yield

$$\frac{\sigma^{(j)}(z)}{\sigma(z)} \equiv F_{m,j}(z) \quad \text{and} \quad \frac{\tau^{(j)}(z)}{\tau(z)} \equiv F_{m,j}^*(z), \quad (2.59)$$

for each z in U_0 and $j=0, 1, 2, \dots$. We use (2.5), (2.59), and (1.18) to see that the substitution $y = \sigma(z)u$, for z in U_0 , transforms (1.1) into

$$u^{(m)} + \sum_{j=1}^m G_{m,j}(z) u^{(m-j)} = 0. \quad (2.60)$$

Similarly, the substitution $v = \tau(z)w$, for z in U_0 , transforms (1.5) into

$$w^{(m)} + \sum_{j=1}^m G_{m,j}^*(z) w^{(m-j)} = 0. \quad (2.61)$$

Of course, $y = \rho(z)v$ from (1.3) transforms (1.1) into (1.5). Therefore, the substitution $u = \chi(z)w$ having $\chi(z) \equiv \rho(z) \tau(z)/\sigma(z)$, for z in U_0 , transforms (2.60) into (2.61). But, (1.18), (1.7), (1.16), and (1.33) yield $G_{m,1}(z) \equiv 0$ and $G_{m,1}^*(z) \equiv 0$. Consequently, $\chi(z)$ is a nonzero constant function on U_0 and $u = \chi(z)w$ transforms (2.60) into

$$w^{(m)} + \sum_{j=1}^m G_{m,j}(z) w^{(m-j)} = 0. \quad (2.62)$$

Since (2.61) and (2.62) coincide, this yields (2.58) and completes the alternate proof.

Of course, (1.18) and (2.58) show that $G_{m,2}(z), \dots, G_{m,m}(z)$ specify $m-1$ algebraically independent semi-invariants of the first kind. For reference in Proposition 2.14, we note that (1.18) yields

$$G_{m,2}(z) \equiv c_2(z) + \left(\frac{1-m}{2}\right) c_1'(z) + \left(\frac{1-m}{2m}\right) (c_1(z))^2. \quad (2.63)$$

Semi-invariants of the first kind corresponding to $G_{m,2}(z)$, $G_{m,3}(z)$ when $m \geq 3$, and $G_{m,4}(z)$ when $m \geq 4$ appeared in [13, Eqs. (11), (12); 14, Eq. (23)]. Formula (2.41) shows that $\beta(z)$ in (1.13) specifies a semi-invariant of the second kind having weight 2. This was noted in [15, p. 446]. For a synopsis of [13–15], see [7, pp. 27–35]. To obtain more useful semi-invariants of the first kind by modifying the relations (2.55), we were led to (1.19) and the following particularly significant new result.

THEOREM 2.13. *On U , (1.19) and (1.33) yield*

$$H_{m,j}^*(z) \equiv H_{m,j}(z), \quad \text{for } j = 1, 2, \dots, m. \quad (2.64)$$

And, when $m \geq 3$, $m-2$ algebraically independent semi-invariants of the first kind are specified by $H_{m,3}(z)$, ..., $H_{m,m}(z)$.

Proof. We use (1.33), (2.55), (2.57), and (1.19) to obtain

$$\begin{aligned} H_{m,j}^*(z) &\equiv \sum_{k=3}^j B_{m,k}^*(z) \binom{m-k}{m-j} F_{m,j-k}^*(z) \\ &\equiv \sum_{k=3}^j \left(\sum_{i=3}^k B_{m,i}(z) \binom{m-i}{m-k} \frac{\rho^{(k-i)}(z)}{\rho(z)} \right) \binom{m-k}{m-j} F_{m,j-k}^*(z) \\ &\equiv \sum_{i=3}^j B_{m,i}(z) \binom{m-i}{m-j} \sum_{k=1}^j \binom{j-i}{j-k} \frac{\rho^{(k-i)}(z)}{\rho(z)} F_{m,j-k}^*(z) \\ &\equiv \sum_{i=3}^j B_{m,i}(z) \binom{m-i}{m-j} \sum_{v=0}^{j-i} \binom{j-i}{v} \frac{\rho^{(v)}(z)}{\rho(z)} F_{m,j-i-v}^*(z) \\ &\equiv \sum_{i=3}^j B_{m,i}(z) \binom{m-i}{m-j} F_{m,j-i}(z) \\ &\equiv H_{m,j}(z), \end{aligned}$$

on U for $j = 1, 2, \dots, m$.

Suppose $m \geq 3$. The $\xi_{m,j}(z)$ in (1.11) and the $F_{m,j}(z)$ in (1.16), (1.17) do not involve $c_3(z)$, ..., $c_m(z)$. And, we have $F_{m,0}(z) \equiv 1$. Therefore, for $j = 3, \dots, m$, $B_{m,j}(z)$ in (1.15) and $H_{m,j}(z)$ in (1.19) effectively involve $c_j(z)$ but do not involve any coefficient of (1.1) having a subscript greater than j . This yields the last assertion of Theorem 2.13 and completes its proof.

PROPOSITION 2.14. *When $m \geq 3$, a relative invariant for (1.1) of weight 3 is specified by*

$$I_{m,3}(z) \equiv H_{m,3}(z). \quad (2.65)$$

When $m \geq 4$, a relative invariant for (1.1) of weight 4 is specified by

$$I_{m,4}(z) \equiv H_{m,4}(z) - \left(\frac{m-3}{2}\right) (I'_{m,3}(z)). \quad (2.66)$$

When $m \geq 5$, a relative invariant for (1.1) of weight 5 is specified by

$$\begin{aligned} I_{m,5}(z) \equiv & H_{m,5}(z) - \left(\frac{m-4}{2}\right) (I'_{m,4}(z)) - \left(\frac{(m-4)(m-3)}{7}\right) (I''_{m,3}(z)) \\ & - \left(\frac{(m-4)(m-3)(7m+13)}{7(m+1)m(m-1)}\right) (G_{m,2}(z) I_{m,3}(z)). \end{aligned} \quad (2.67)$$

Moreover, when $m \geq 4$, the relative invariants corresponding to (2.65) and (2.66) are algebraically independent; and, when $m \geq 5$, the relative invariants corresponding to (2.65), (2.66), and (2.67) are algebraically independent.

Proof. We deduce $H_{m,3}(z) \equiv B_{m,3}(z)$ from (1.19) and (1.16). Thus, Corollary 2.10 shows that (2.65) specifies a relative invariant of weight 3.

Suppose $m \geq 4$. Let λ be a complex number and let $\Phi(z)$ on Ω be the polynomial combination of (1.25) defined by

$$\Phi(z) \equiv H_{m,4}(z) + \lambda I'_{m,3}(z). \quad (2.68)$$

Let $\Phi^*(z)$ on U and $\Phi^{**}(\zeta)$ on V be the corresponding polynomial combinations of (1.26) and (1.27). Theorem 2.13 and (2.65) yield $\Phi^*(z) \equiv \Phi(z)$ on U . Using (2.56), (2.34), and (2.35), we deduce

$$\begin{aligned} \Phi^{**}(\zeta) &\equiv H_{m,4}^{**}(\zeta) + \lambda (D_{\zeta} I_{m,3}^{**}(\zeta)) \\ &\equiv H_{m,4}^{**}(\zeta) + \lambda \{D_{\zeta} [(I_{m,3}(f(\zeta)))(f'(\zeta))^3]\} \\ &\equiv (\Phi^*(f(\zeta)))(f'(\zeta))^4 + \frac{3}{2}(2\lambda + (m-3))(f'(\zeta))^2 (f''(\zeta))(I_{m,3}(f(\zeta))), \end{aligned}$$

on V . Thus, $\Phi(z)$ in (2.68) specifies a relative invariant of weight 4 if and only if $\lambda = -(m-3)/2$; and, (2.66) specifies a relative invariant of weight 4.

Suppose $m \geq 5$. Let $\lambda_1, \lambda_2, \lambda_3$ be complex numbers and let $\Psi(z)$ on Ω be the polynomial combination of (1.25) defined by

$$\Psi(z) \equiv H_{m,5}(z) + \lambda_1 I'_{m,4}(z) + \lambda_2 I''_{m,3}(z) + \lambda_3 G_{m,2}(z) I_{m,3}(z). \quad (2.69)$$

Let $\Psi^*(z)$ on U and $\Psi^{**}(\zeta)$ on V be the corresponding polynomial combinations of (1.26) and (1.27). Our results about (2.64), (2.65), (2.66), and (2.58) yield $\Psi^*(z) \equiv \Psi(z)$ on U . For $G_{m,2}^*(z)$ in (1.33) and $G_{m,2}^{**}(\zeta)$ in (1.34)

corresponding to $G_{m,2}(z)$ in (1.32) and (2.63), we apply (2.44) and (2.43) to obtain

$$G_{m,2}^{**}(\zeta) \equiv (G_{m,2}^*(f(\zeta)))(f'(\zeta))^2 + \binom{m+1}{3} \left(\left(\frac{f'''(\zeta)}{2f'(\zeta)} \right) - \left(\frac{3}{4} \right) \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2 \right), \quad (2.70)$$

on V . Using (2.56), (2.34), (2.35), (2.36), (2.66), (2.65), and (2.70), we deduce, for ζ in V , that

$$\begin{aligned} \Psi^{**}(\zeta) &\equiv H_{m,5}^{**}(\zeta) + \lambda_1(D_\zeta I_{m,4}^{**}(\zeta)) + \lambda_2(D_\zeta^2 I_{m,3}^{**}(\zeta)) + \lambda_3(G_{m,2}^{**}(\zeta) I_{m,3}^{**}(\zeta)) \\ &\equiv (\Psi^*(f(\zeta)))(f'(\zeta))^5 + \chi_1(\zeta) + \chi_2(\zeta) + \chi_3(\zeta) + \chi_4(\zeta), \end{aligned}$$

where

$$\begin{aligned} \chi_1(\zeta) &\equiv (4\lambda_1 + 2(m-4))(f'(\zeta))^3 (f''(\zeta))(I_{m,4}(f(\zeta))), \\ \chi_2(\zeta) &\equiv (7\lambda_2 + (m-4)(m-3))(f'(\zeta))^3 (f''(\zeta))(I'_{m,3}(f(\zeta))), \\ \chi_3(\zeta) &\equiv \left(\frac{3}{4} \right) \left(8\lambda_2 - \binom{m+1}{3} \lambda_3 - \binom{m-3}{3} \right) \\ &\quad \times (f'(\zeta))(f''(\zeta))^2 (I_{m,3}(f(\zeta))) \\ \chi_4(\zeta) &\equiv \left(\frac{1}{6} \right) \left(18\lambda_2 + 3 \binom{m+1}{3} \lambda_3 + (m+7) \binom{m-3}{2} \right) \\ &\quad \times (f'(\zeta))^2 (f'''(\zeta))(I_{m,3}(f(\zeta))). \end{aligned}$$

We find the condition $\chi_1(\zeta) \equiv \chi_2(\zeta) \equiv \chi_3(\zeta) \equiv 0$ is satisfied if and only if $\lambda_1, \lambda_2, \lambda_3$ for (2.69) have the values indicated in (2.67); and for these values we also have $\chi_4(\zeta) \equiv 0$. Thus, (2.67) specifies a relative invariant of weight 5.

For $3 \leq j \leq 5$ and $j \leq m$, $I_{m,j}(z)$ effectively involves $c_j(z)$ but does not involve any coefficient of (1.1) having a subscript greater than j . This yields the last assertion in Proposition 2.14 and completes its proof.

The $m \times m$ matrices

$$X(\zeta) \equiv (P_m(f(\zeta)))(R_m(\zeta))$$

and

$$Y(\zeta) \equiv (Q_m(f(\zeta)))(R_m(\zeta)) + S_m(\zeta),$$

defined through (2.6) and (2.38) for each ζ in V , depend only on (1.2) and m . They satisfy $\det(X(\zeta)) \neq 0$, for each ζ in V . When (1.2) and m are fixed, each (1.1) on a region Ω containing U specifies corresponding companion matrices $M_m(z)$ and $M_m^{**}(\zeta)$ for (2.2) and (2.4).

PROPOSITION 2.15. *Relative to $m \geq 2$ and a given substitution (1.2), $X(\zeta)$ and $Y(\zeta)$ are unique $m \times m$ matrices over E_V such that*

$$M_m^{**}(\zeta) \equiv (M_m(f(\zeta)))(X(\zeta)) + Y(\zeta), \quad (2.71)$$

for each (1.1) on a region Ω containing U and each ζ in V . Moreover, the corresponding $J_m(z)$ in (2.21) and $J_m^{**}(\zeta)$ in (2.52) are related by

$$J_m^{**}(\zeta) \equiv (J_m(f(\zeta)))(X(\zeta)), \quad (2.72)$$

for each ζ in V .

Proof. Propositions 2.2, 2.7 and Corollaries 2.4, 2.9 yield (2.71) and (2.72). To establish the uniqueness of $X(\zeta)$ and $Y(\zeta)$, suppose $X_1(\zeta)$ and $Y_1(\zeta)$ are $m \times m$ matrices over E_V that satisfy

$$M_m^{**}(\zeta) \equiv (M_m(f(\zeta)))(X_1(\zeta)) + Y_1(\zeta)$$

and therefore $(M_m(f(\zeta)))(X_1(\zeta) - X(\zeta)) \equiv Y(\zeta) - Y_1(\zeta)$, for each (1.1) on an Ω containing U and each ζ in V . From this we deduce $X_1(\zeta) \equiv X(\zeta)$ and $Y_1(\zeta) \equiv Y(\zeta)$ to complete the proof.

PROPOSITION 2.16. *The homogeneous linear differential equation (1.1), for $m \geq 2$, has a fundamental system of local solutions of the form (1.10) if and only if there is a substitution (1.2) such that the corresponding (1.6) satisfies*

$$c_j^{**}(\zeta) \equiv 0, \quad \text{for } j = 1, 2, \dots, m. \quad (2.73)$$

Proof. Suppose $\phi(z)$, $\psi(z)$ are meromorphic functions on a subregion U_0 of Ω such that (1.10) is a fundamental system of solutions for (1.1) on U_0 . Then, there is a subregion U of U_0 such that $\phi(z) \neq 0$, for each z in U , and the function g defined on U by

$$\zeta = g(z) = \psi(z)/\phi(z), \quad \text{for each } z \text{ in } U,$$

is a univalent analytic function on U . Let (1.2) be the substitution having $\rho(z) \equiv (\phi(z))^{m-1}$ on U and $z = f(\zeta) = g^{-1}(\zeta)$ on $V = g(U)$. Then, the substitution $y = \rho(z)v$ transforms (1.1) into a (1.5) having

$$\{1, g(z), (g(z))^2, \dots, (g(z))^{m-1}\} \quad (2.74)$$

as a fundamental system of solutions on U . In this (1.5), we set $z = f(\zeta)$ and $t(\zeta) = v(f(\zeta))$ to obtain a (1.6) having

$$\{1, \zeta, \zeta^2, \dots, \zeta^{m-1}\} \quad (2.75)$$

as a fundamental system of solutions on V . Therefore, this (1.6) satisfies (2.73).

Suppose a substitution (1.2) transforms (1.1) into some (1.6) that satisfies (2.73). Set $\zeta = g(z) = f^{-1}(z)$ on U . There is an analytic function $\phi(z)$ on a subregion U_0 of U such that $(\phi(z))^{m-1} \equiv \rho(z)$ on U_0 . Set $\psi(z) \equiv g(z)\phi(z)$ on U_0 . Since (2.73) shows that (2.75) is a fundamental system of solutions on $V_0 = g(U_0)$ for this (1.6), (2.74) is a fundamental system of solutions on U_0 for the corresponding (1.5) and (1.10) is a fundamental system of solutions on U_0 for (1.1). This completes the proof.

THEOREM 2.17. *The homogeneous linear differential equation (1.1), for $m \geq 2$, has a fundamental system of local solutions of the form (1.10) if and only if*

$$B_{m,j}(z) \equiv 0, \quad \text{for } j = 1, 2, \dots, m. \quad (2.76)$$

Proof. Suppose (1.1) satisfies (2.76). Then, (1.15) and Proposition 4.4 show that (1.1) coincides with (1.9) and therefore has a fundamental system of local solutions of the form (1.10).

Suppose (1.1) has a fundamental system of local solutions of the form (1.10). Then, Proposition 2.16 yields a substitution (1.2) that transforms (1.1) into some (1.6) satisfying (2.73). For this (1.6), we have $\alpha^{**}(\zeta) \equiv 0$, $\beta^{**}(\zeta) \equiv 0$, $\xi_{m,j}^{**}(\zeta) \equiv 0$, for $j = 1, 2, \dots, m$, and

$$B_{m,j}^{**}(\zeta) \equiv c_j^{**}(\zeta) - \xi_{m,j}^{**}(\zeta) \equiv 0, \quad \text{for } j = 1, 2, \dots, m. \quad (2.77)$$

Since the components of $J_m^{**}(\zeta)$ in (2.52) satisfy (2.77), (2.72) shows that the components of $J_m(z)$ in (2.21) satisfy (2.76). This completes the proof.

COROLLARY 2.18. *For $m \geq 2$, there is a substitution (1.2) that transforms (1.1) into some (1.6) satisfying (2.73) if and only if (1.1) satisfies (2.76).*

Proof. Proposition 2.16 and Theorem 2.17 yield this result.

COROLLARY 2.19. *For $m \geq 2$, suppose (1.1) has a fundamental system of local solutions of the form (1.10) and suppose $d_1(z), \dots, d_m(z)$ are meromorphic functions on Ω satisfying*

$$d_1(z) \equiv c_1(z), \quad d_2(z) \equiv c_2(z) \quad (2.78)$$

and the requirement that the differential equation

$$y^{(m)} + \sum_{j=1}^m d_j(z) y^{(m-j)} = 0 \quad (2.79)$$

has a fundamental system of local solutions of the form

$$\{(\sigma(z))^{m-1-i}(\tau(z))^i: \text{for } i = 0, 1, \dots, m-1\}.$$

Then, $d_j(z) \equiv c_j(z)$, for $j = 1, 2, \dots, m$.

Proof. Due to (2.78), the elements $\xi_{m,1}(z), \dots, \xi_{m,m}(z)$ defined for (1.1) by means of (1.12)–(1.14) and (1.11) are the same as the corresponding elements defined for (2.79). Therefore, Theorem 2.17 yields

$$d_j(z) \equiv \xi_{m,j}(z) \equiv c_j(z), \quad \text{for } j = 1, 2, \dots, m.$$

This completes the proof.

COROLLARY 2.20. *For a given (1.1) having $m \geq 2$, there are unique meromorphic functions $\xi_{m,1}(z), \dots, \xi_{m,m}(z)$ on Ω such that (1.8) is satisfied and (1.9) has a fundamental system of local solutions of the form (1.10). Moreover, they are given by (1.11)–(1.14).*

Proof. Proposition 4.4 shows that (1.11)–(1.14) specify functions having the indicated properties. To establish their uniqueness, we apply Corollary 2.19 with (1.9) in place of (1.1). This completes the proof.

3. ABSENCE OF MOVABLE BRANCH POINTS

Let the coefficients $a(z), \dots, f(z)$ of

$$\begin{aligned} a(z) y''^2 + b(z) y'' y' + c(z) y'' y + d(z) y'^2 \\ + e(z) y' y + f(z) y^2 = 0 \end{aligned} \quad (3.1)$$

belong to the field E_Ω of meromorphic functions of a complex variable z on a region Ω of the complex plane, let $Q(z, y, y', y'')$ denote the left member of (3.1), and set

$$\Delta(z) \equiv \begin{vmatrix} a(z) & b(z)/2 & c(z)/2 \\ b(z)/2 & d(z) & e(z)/2 \\ c(z)/2 & e(z)/2 & f(z) \end{vmatrix}.$$

To specify all such $Q(z, y, y', y'')$ that satisfy important identities of the type (3.21), we first need a precise characterization of the various (3.1)'s whose solutions are free of movable branch points.

When $a(z) \neq 0 \neq \Delta(z)$, (3.1) can be written as

$$\begin{aligned} (2a(z) y'' + b(z) y' + c(z) y)^2 + A_2(z) y'^2 \\ + A_3(z) y' y + A_4(z) y^2 = 0, \end{aligned} \quad (3.2)$$

where

$$A_2 \equiv 4ad - b^2, \quad A_3 \equiv 4ae - 2bc, \quad A_4 \equiv 4af - c^2,$$

and where the expression $D(z) \equiv (A_3(z))^2 - 4(A_2(z))(A_4(z))$ satisfies

$$D(z) \equiv -2^6(a(z))(\Delta(z)) \neq 0.$$

We established in [11] that, when $a(z) \neq 0 \neq \Delta(z)$, the solutions of (3.1) are free of movable branch points in the precise sense of [5, 11] if and only if $A_2(z) \neq 0$ and (3.2) satisfies

$$b(z) \equiv \left(\frac{2A'_2(z) + 2A_3(z)}{A_2(z)} - \frac{D'(z)}{D(z)} \right) (a(z)) \quad (3.3)$$

and

$$c(z) \equiv \left(\frac{A'_3(z) + 2A_4(z)}{A_2(z)} - \frac{A_3(z) D'(z)}{2A_2(z) D(z)} \right) (a(z)). \quad (3.4)$$

THEOREM 3.1. *The solutions of (3.1) are free of movable branch points in the precise sense of [5, 11] if and only if*

$$\Delta(z) \equiv 0 \quad \text{and} \quad Q(z, y, y', y'') \neq 0 \quad (3.5)$$

or

$$(a(z))(A_2(z))(\Delta(z)) \neq 0 \quad \text{and} \quad (3.2) \text{ satisfies (3.3) and (3.4)} \quad (3.6)$$

or

$$a(z) \equiv b(z) \equiv 0 \neq c(z) \quad \text{and} \quad d(z) \equiv -c(z) \quad (3.7)$$

or

$$a(z) \equiv b(z) \equiv 0 \neq c(z)$$

and

$$d(z) \equiv \left(\frac{1-m}{m} \right) (c(z)), \quad \text{for some integer } m \neq 0, 1. \quad (3.8)$$

Moreover, if (3.5) or (3.6) or (3.7) or (3.8) for $m > 1$ is satisfied, then the solutions of (3.1) are free of movable singularities. And, if (3.8) for $m < 0$ is satisfied, then the solutions of (3.1) have movable poles but no movable essential singularities.

Explanation. Both (3.7) and (3.8) are consistent with [38, 39, 23, 29, 10]. However, because we shall base Theorem 3.1 on precise definitions, the α -theory in [38, 39, 23, 29, 10] is not directly applicable.

Let $S_{br}(Q)$ denote the subset of Ω such that, for any z_0 in Ω , z_0 belongs to $S_{br}(Q)$ if and only if z_0 is a branch point for some local solution of (3.1). To say that z_0 is a *movable branch point for the solutions of (3.1) relative to Ω* means that z_0 is an interior point of $S_{br}(Q)$. Similarly, subsets $S_s(Q)$, $S_{po}(Q)$, and $S_{ess}(Q)$ of Ω are defined and their respective interior points are the *movable singularities*, *movable poles*, and *movable essential singularities* for the solutions of (3.1) relative to Ω . Namely, for z_0 in Ω , z_0 belongs to $S_s(Q)$ or to $S_{po}(Q)$ or to $S_{ess}(Q)$, respectively, according to whether z_0 is a singular point or a pole or an essential singular point for some local solution of (3.1). For z_0 in Ω , to say z_0 is a *singular point for some local solution $y_0(z)$ of (3.1) relative to analytic continuation in Ω* means there is a curve γ in Ω from the subregion U of Ω on which $y_0(z)$ is defined to z_0 such that $y_0(z)$ can be analytically continued from U along γ to any point of γ except z_0 but cannot be analytically continued along γ to include z_0 . In particular, $S_s(Q)$ contains $S_{br}(Q)$, $S_{po}(Q)$, and $S_{ess}(Q)$.

Naturally, to say *the solutions of (3.1) relative to Ω are free of movable branch points*, or *are free of movable singularities*, or *are free of movable poles*, or *are free of movable essential singularities* means the corresponding set $S_{br}(Q)$, or $S_s(Q)$, or $S_{po}(Q)$, or $S_{ess}(Q)$ has no interior points. As in [11, pp. 82–83], the dependence of Theorem 3.1 on Ω can be removed by observing that each of the relations (3.5), (3.6), (3.7), (3.8) is preserved under simultaneous analytic continuation of $a(z)$, ..., $f(z)$.

Proof of Theorem 3.1. If $a(z) \not\equiv 0 \equiv \Delta(z)$ or

$$a(z) \equiv b(z) \equiv c(z) \equiv 0 \not\equiv Q(z, y, y', y'').$$

then $Q(z, y, y', y'')$ is expressible as a product of nonzero homogeneous linear polynomials in y, y', y'' over some quadratic extension field of E_Ω (e.g., see [11, p. 86]). When $a(z) \equiv 0$ and either $b(z) \not\equiv 0$ or $c(z) \not\equiv 0$, we find the condition $\Delta(z) \equiv 0$ is satisfied if and only if $Q(z, y, y', y'')$ is expressible as a product of $(b(z)y' + c(z)y)$ and some nonzero homogeneous linear polynomial in y, y', y'' over E_Ω . Thus, whenever (3.5) is satisfied, the solutions of (3.1) are free of movable singularities. Due to the information about (3.6) in [11], it merely remains for us to examine the case $a(z) \equiv 0 \not\equiv \Delta(z)$.

Suppose (3.1) satisfies $a(z) \equiv 0 \not\equiv \Delta(z)$ and its solutions are free of movable branch points. Under the substitution

$$y' = yw, \quad y'' = y(w' + w^2), \quad (3.9)$$

(3.1) is transformed into

$$(b(z)w + c(z))w' + b(z)w^3 + (c(z) + d(z))w^2 + e(z)w + f(z) = 0. \quad (3.10)$$

As a consequence of Lemma 3.2, the solutions of (3.10) are free of movable branch points. Therefore, the situation $n=1$ of [11, Theorem 1.1] shows that $(b(z)w + c(z))$ divides the left member of (3.10). Due to $d(z) \neq 0$, this requires $b(z) \equiv 0$, $c(z) \neq 0$, and $d(z) \neq 0$. If $d(z) \equiv -c(z)$, then we have (3.7). Thus, suppose $d(z) \neq -c(z)$. To obtain (3.8), we rewrite (3.10) as

$$w' + \rho(z)w^2 + \sigma(z)w + \tau(z) = 0, \quad (3.11)$$

where $\rho(z) \equiv 1 + (d(z)/c(z))$, $\sigma(z) \equiv e(z)/c(z)$, and $\tau(z) \equiv f(z)/c(z)$. There is a simply connected subregion U of Ω and an analytic function $F(z)$ on U such that: $\rho(z)$, $\sigma(z)$, $\tau(z)$ are analytic on U ; $0 \neq \rho(z) \neq 1$, for each z in U ; and the function $w_0(z) \equiv F'(z)$ is a solution of (3.11) on U . We obtain

$$u' - (2w_0(z)\rho(z) + \sigma(z))u - \rho(z) = 0 \quad (3.12)$$

from (3.11) by means of $w = w_0 + (1/u)$. Let $u_0(z)$ be an analytic solution of (3.12) on U . We obtain

$$v' - (2w_0(z)\rho(z) + \sigma(z))v = 0 \quad (3.13)$$

from (3.12) by means of $u = u_0 + v$. There is a simply connected subregion V of U and there is an analytic solution $v_0(z)$ of (3.13) on V such that $v_0(z) \neq 0$, for each z in V , and the analytic function ϕ defined on V by

$$\phi(z) \equiv (u_0(z))/(v_0(z)), \quad \text{for each } z \text{ in } V,$$

is univalent on V . Because $u_0(z)$ and $v_0(z)$ are solutions on V of (3.12) and (3.13), respectively, we obtain

$$\phi'(z) \equiv (\rho(z))/(v_0(z)), \quad \text{for each } z \text{ in } V.$$

Therefore, a meromorphic solution of (3.11) on V is given by

$$w(z) = F'(z) + \frac{1}{u_0(z) - Kv_0(z)} = F'(z) + \frac{\phi'(z)}{\rho(z)(\phi(z) - K)},$$

for each constant K . Let C , C_1 , C_2 be circles of positive radii contained in V such that C_1 and C_2 are disjoint, the center a of C_1 lies on C , and C_2

is inside C . Then, for any point s inside C_2 , we use (3.9) to see that an analytic solution $y_s(z)$ of (3.1) is defined for z inside C_1 by

$$y_s(z) = (\exp(F(z))) \exp \left(\int_{[a,z]} \frac{\phi'(\zeta) d\zeta}{\rho(\zeta)(\phi(\zeta) - \phi(s))} \right),$$

where $[a, z]$ denotes the line segment from a to z . Moreover, because ϕ is univalent on V , $y_s(z)$ can be analytically continued along any curve in V from a that does not contain s .

To show that $1/\rho(s)$ is an integer for each s inside C_2 , suppose s_0 is a point inside C_2 and $1/\rho(s_0)$ is not an integer. Then, there is a neighborhood $N(s_0)$ of s_0 inside C_2 such that, for each s in $N(s_0)$, $1/\rho(s)$ is not an integer. For s fixed in $N(s_0)$, the univalence of ϕ on V shows that

$$\frac{\phi'(\zeta)}{\rho(\zeta)(\phi(\zeta) - \phi(s))} - \frac{\phi'(\zeta)}{\rho(s)(\phi(\zeta) - \phi(s))}$$

has a finite limit as ζ approaches s and defines through continuity an analytic function of ζ on V . Therefore, the integrals

$$I_1 = \int_C \frac{\phi'(\zeta) d\zeta}{\rho(\zeta)(\phi(\zeta) - \phi(s))} \quad \text{and} \quad I_2 = \frac{1}{\rho(s)} \int_C \frac{\phi'(\zeta) d\zeta}{\phi(\zeta) - \phi(s)}$$

taken one time counterclockwise about the circle C satisfy $I_1 = I_2$. But, the only zero of $\phi(\zeta) - \phi(s)$ on V is the simple zero $\zeta = s$. Therefore, we obtain $\rho(s)I_2 = 2\pi i$ by the argument principle (e.g., see [1, pp. 138–139]). This yields $\exp(I_1) \neq 1$. When $y_s(z)$ on the inside of C_1 is analytically continued one time counterclockwise about C to obtain a function $\hat{y}_s(z)$ defined on the inside of C_1 , we have

$$\hat{y}_s(z) = (\exp(I_1))(y_s(z)) \neq y_s(z), \quad \text{for each } z \text{ inside } C_1.$$

Hence, for each s in $N(s_0)$, s is a branch point for the corresponding local solution $y_s(z)$ of (3.1). This contradiction to the absence of movable branch points for the solutions of (3.1) shows that $1/\rho(s)$ is an integer when s is a point inside C_2 . Continuity of $\rho(z)$ inside C_2 yields an integer $m \neq 0, 1$ such that $\rho(z) = 1/m$, for each z in Ω . Thus, we obtain (3.8).

To complete the proof of Theorem 3.1, we apply Propositions 3.3 and 3.4.

LEMMA 3.2. *If the solutions of (3.1) are free of movable branch points and if the substitution (3.9) is used to transform (3.1) into*

$$P(z, w, w') = 0 \tag{3.14}$$

according to $Q(z, y, yw, y(w' + w^2)) \equiv y^2 P(z, w, w')$, then the solutions of (3.14) are free of movable branch points.

Proof. Let $S_{br}(P)$ be defined for (3.14) just as $S_{br}(Q)$ was defined for (3.1). Suppose z_0 belongs to $S_{br}(P)$. Then, there is a deleted neighborhood

$$N_*(z_0, \delta) = \{z: 0 < |z - z_0| < \delta\}$$

of z_0 and there is an analytic function $G(z)$ on an open disk D contained in $N_*(z_0, \delta)$ such that: $w_0(z) \equiv G'(z)$ is a local solution of (3.14) on D ; $w_0(z)$ can be analytically continued along any curve from D contained in $N_*(z_0, \delta)$; and analytic continuation of $w_0(z)$ from D one time about z_0 and back to D does not return it to its initial branch on D . Thus, the function

$$y_0(z) = \exp(G(z)), \quad \text{for } z \text{ in } D,$$

is a local solution of (3.1) on D and $y_0(z)$ can be analytically continued along any curve from D in $N_*(z_0, \delta)$. Moreover, analytic continuation of $y_0(z)$ from D one time about z_0 and back to D does not return it to its initial branch on D ; otherwise,

$$w_0(z) = y'_0(z)/y_0(z)$$

would return to its initial branch. Thus, z_0 belongs to $S_{br}(Q)$. Since $S_{br}(P)$ is a subset of $S_{br}(Q)$, neither contains interior points. This completes the proof.

The proof of [11, Proposition 6.4] implicitly used the contrapositive of Lemma 3.2.

PROPOSITION 3.3. *Suppose $\alpha(z)$, $\beta(z)$ belong to E_Ω and let U be any simply connected subregion of Ω that does not contain poles of $\alpha(z)$ or $\beta(z)$. Then, the solutions of*

$$y''y - y'^2 + \alpha(z)y'y + \beta(z)y^2 = 0 \quad (3.15)$$

are free of movable singularities and there is a noncountably infinite linearly independent set of analytic solutions on U of (3.15).

Proof. There are analytic functions $F(z)$, $G(z)$ on U such that $F'(z) \not\equiv 0$ and the solutions of

$$w' + \alpha(z)w + \beta(z) = 0 \quad (3.16)$$

on U are $w(z) = K_1 F'(z) + G'(z)$, where K_1 is an arbitrary constant. That

(3.9) relates (3.15) and (3.16) was noted in [39, p. 38, Eq. (6)]. Thus, the solutions of (3.15) on U are

$$y(z) = K_2 \exp(K_1 F(z) + G(z)), \quad (3.17)$$

where K_1, K_2 are arbitrary constants. Our freedom in selecting U shows that the only singular points that local solutions of (3.15) may have occur at the poles of $\alpha(z)$ or $\beta(z)$. Thus, the solutions of (3.15) are free of movable singularities. Of course, the set of local solutions of (3.15) given by (3.17) for $K_2 = 1$ and arbitrary K_1 is noncountably infinite. It is also linearly independent by Proposition 5.3. This completes the proof.

PROPOSITION 3.4. *Suppose $\alpha(z), \beta(z)$ belong to E_Ω , let U be any simply connected subregion of Ω that does not contain any poles of $\alpha(z)$ or $\beta(z)$, and let m be a nonzero integer. When $m > 0$, the solutions of*

$$y''y + \left(\frac{1-m}{m}\right)y'^2 + \alpha(z)y'y + m\beta(z)y^2 = 0 \quad (3.18)$$

are free of movable singularities. When $m < 0$, the solutions of (3.18) are free of movable branch points and movable essential singularities, but have movable poles. And, when $m < 0$, there is an infinite linearly independent set of meromorphic solutions on U of (3.18).

Proof. Let $\{\phi(z), \psi(z)\}$ be a fundamental system of analytic solutions on U of

$$u'' + \alpha(z)u' + \beta(z)u = 0 \quad (3.19)$$

(e.g., see [1, pp. 308–311, 295–296]). That the substitution $y = u^m$ relates (3.18) and (3.19) was noted in [39, p. 38, Eq. (1)]. Thus, the nonzero solutions on U of (3.18) are

$$y(z) = (C_1\phi(z) + C_2\psi(z))^m, \quad (3.20)$$

where C_1, C_2 are constants not both zero. We use (3.20) and our freedom in selecting U to see that any point in $S_{\text{br}}(Q)$ or $S_{\text{ess}}(Q)$ must be a pole of $\alpha(z)$ or $\beta(z)$. Thus, neither $S_{\text{br}}(Q)$ nor $S_{\text{ess}}(Q)$ has interior points. Similarly, when $m > 0$, each point of $S_s(Q)$ is a pole of $\alpha(z)$ or $\beta(z)$ and is not an interior point. When $m < 0$, S_{po} contains U because each z_0 in U specifies constants $C_1 = \psi(z_0)$ and $C_2 = -\phi(z_0)$, not both zero, such that the corresponding (3.20) has z_0 as a pole. And, when $m < 0$, the set of meromorphic solutions

$$y_k(z) = (\phi(z) + k\psi(z))^m, \quad \text{for } k = 0, 1, 2, \dots,$$

on U of (3.18) is infinite and linearly independent by Theorem 5.1. This completes the proof.

PROPOSITION 3.5. *Suppose the left member Q of (3.1) satisfies $\Delta(z) \neq 0$ and*

$$\sum_{i=0}^k (P_i)(Q^{(i)}) \equiv \prod_{j=1}^r (L_j)^{v_j} \neq 0, \quad (3.21)$$

where k, r, v_1, \dots, v_r are positive integers, P_0, \dots, P_k are polynomials in y, y', \dots over E_Ω , and L_1, \dots, L_r are homogeneous linear polynomials in y, y', \dots over some algebraic extension field of E_Ω . Then, either Q satisfies (3.6) or Q satisfies (3.8) with $m \geq 2$.

Proof. Let D be a subregion of Ω on which the restriction of (3.21) can be rewritten so that all the coefficients belong to the field E_D of meromorphic functions defined on D . Then, each local solution of $Q=0$ on a subregion of D is a local solution of at least one of $L_1=0, \dots, L_r=0$. Since $L_1=0, \dots, L_r=0$ are homogeneous linear differential equations finite in number, the solutions of $Q=0$ relative to D are free of movable branch points. Applying Propositions 3.3 and 3.4 with Ω replaced by D , we see that Q cannot satisfy (3.7) or (3.8) with $m < 0$ because none of $L_1=0, \dots, L_r=0$ can possess an infinite linearly independent set of solutions. Applying Theorem 3.1 with Ω replaced by D , we find that either (3.6) or (3.8) for $m \geq 2$ is satisfied on D and therefore also on Ω . This completes the proof.

Assuming $a(z) \neq 0 \neq \Delta(z)$, we established in [11] that (3.1) satisfies (3.6) if and only if

$$\begin{aligned} (Q(z, y, y', y''))' + \lambda(z) Q(z, y, y', y'') \\ \equiv (M(z, y, y', y''))(N(z, y, y', y'', y''')), \end{aligned}$$

for some $\lambda(z)$ in E_Ω and some homogeneous linear polynomials M, N in y, y', y'', y''' over E_Ω .

PROPOSITION 3.6. *Suppose (3.1) satisfies $a(z) \equiv 0 \neq \Delta(z)$. Then, there is a nontrivial factorization of $(Q(z, y, y', y''))' + \lambda(z) Q(z, y, y', y'')$ for some $\lambda(z)$ in E_Ω if and only if*

$$b(z) \equiv 0, \quad d(z) \equiv \left(\frac{-1}{2} \right) c(z) \neq 0, \quad \lambda(z) \equiv \frac{-c'(z) + 2e(z)}{c(z)}. \quad (3.22)$$

Moreover, for $\alpha(z)$ and $\beta(z)$ in E_Ω , the relation

$$(Q_2)' + (2\alpha(z))(Q_2) \equiv yL_2(z, y, y', y'', y''') \quad (3.23)$$

is satisfied by $Q_2 \equiv y''y + (-1/2)y'^2 + \alpha(z)y'y + 2\beta(z)y^2$ and

$$L_2 \equiv y''' + 3\alpha y'' + (\alpha' + 2\alpha^2 + 4\beta)y' + (2\beta' + 4\alpha\beta)y. \quad (3.24)$$

Proof. For $a(z) \equiv 0$ and for any $\lambda(z)$ in E_Q , we obtain

$$\begin{aligned} Q' + \lambda Q &\equiv (by' + cy)y''' + by''^2 + (b' + \lambda b + c + 2d)y''y' \\ &\quad + (c' + \lambda c + e)y'y + (d' + \lambda d + e)y'^2 \\ &\quad + (e' + \lambda e + 2f)y'y + (f' + \lambda f)y^2. \end{aligned} \quad (3.25)$$

Thus, $(Q' + \lambda Q - (by' + cy)y''')$ is divisible by $(by' + cy)$ if and only if $(Q' + \lambda Q)$ has a nontrivial factorization. Therefore, the latter occurs if and only if (3.22) is satisfied. Substituting $c \equiv 1$, $e \equiv \alpha$, $f \equiv 2\beta$, and (3.22) into (3.25), we obtain (3.23). This completes the proof.

4. REMARKABLE NEW IDENTITIES

THEOREM 4.1. Suppose E is a field having a derivation $'$ and n is a positive integer such that $n! \neq 0$ for $0, 1$ in E . Let α, β be elements in E and set

$$Q_n \equiv y''y + \left(\frac{1-n}{n}\right)y'^2 + \alpha y'y + n\beta y^2. \quad (4.1)$$

Then, there are polynomials

$$P_{n,n-1}(y), \quad P_{n,n-2}(y, y'), \dots, \quad P_{n,0}(y, y', \dots, y^{(n-1)}) \quad (4.2)$$

in $y, y', \dots, y^{(n-1)}$ over E such that Q_n satisfies

$$\sum_{j=0}^{n-1} (P_{n,j})(Q_n^{(j)}) \equiv y^n \left(y^{(n+1)} + \sum_{j=0}^n a_{n,j} y^{(j)} \right), \quad (4.3)$$

where $a_{i,j}$, for $i=0, 1, \dots, n$ and $j=-1, 0, 1, \dots, n+1$, is recursively defined in E by

$$\begin{aligned} a_{i,-1} &= 0, & \text{for } i=0, 1, \dots, n; & & a_{0,0} &= 0; \\ a_{i,i+1} &= 1, & \text{for } i=0, 1, \dots, n; & & & \\ a_{i,j} &= 0, & \text{for } i=0, 1, \dots, n-1 \text{ and } i+2 \leq j \leq n+1; & & & \\ a_{1,0} &= n\beta; & a_{1,1} &= \alpha; & & \\ a_{i+1,j} &= a_{i,j-1} + (a_{i,j})' + (i+1)\alpha a_{i,j} + (i+1)(n-i)\beta a_{i-1,j}, & & & & \\ & \text{for } i=1, 2, \dots, n-1 \text{ and } 0 \leq j \leq i+1. & & & & \end{aligned} \quad (4.4)$$

Proof. Let $\{y^{(0)}, w^{(0)}, w^{(1)}, w^{(2)}, w^{(3)}, \dots\}$ be an algebraically independent set of variables over E and let \mathcal{R} denote the ring of polynomials in $y^{(0)}, w^{(0)}, w^{(1)}, \dots$ over E . Then, there is a unique way to extend the derivation $'$ of E to a derivation $'$ of \mathcal{R} such that

$$(y^{(0)})' = y^{(0)}w^{(0)} \quad \text{and} \quad (w^{(i)})' = w^{(i+1)}, \quad \text{for } i = 0, 1, 2, \dots$$

Let V_0, V_1, V_2, \dots and $y^{(1)}, y^{(2)}, y^{(3)}, \dots$ be defined in \mathcal{R} by

$$V_0 = 1 \quad \text{and} \quad V_{i+1} = (V_i)' + (w^{(0)})(V_i), \quad \text{for } i = 0, 1, 2, \dots, \quad (4.5)$$

and

$$y^{(i)} = (y^{(0)})(V_i), \quad \text{for } i = 1, 2, 3, \dots \quad (4.6)$$

The derivation $'$ of \mathcal{R} yields

$$(y^{(0)})' = y^{(0)}w^{(0)} = y^{(0)}V_1 = y^{(1)}$$

and, for $i = 1, 2, \dots$,

$$(y^{(i)})' = (y^{(0)}V_i)' = (y^{(0)})(w^{(0)}V_i + (V_i)') = y^{(i+1)}.$$

Due to [11, Proposition 3.4], the set $\{y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}, \dots\}$ is algebraically independent over E . Therefore, we can identify $y^{(0)}, y^{(1)}, \dots$ with the variables y, y', \dots in the statement of Theorem 4.1. We shall also write w, w', \dots for $w^{(0)}, w^{(1)}, \dots$. Since the ring \mathcal{R} is an integral domain, there is a unique way to extend the derivation $'$ of \mathcal{R} to a derivation $'$ of the quotient field \mathcal{F} of \mathcal{R} (e.g., see [6, p. 45, Proposition 11]).

Relative to Q_n in \mathcal{R} , we introduce in \mathcal{F}

$$H_0 \equiv 0, \quad H_1 \equiv \frac{Q_n}{y^2}; \quad (4.7)$$

and, when $n \geq 2$, we use the derivation $'$ of \mathcal{F} to define in \mathcal{F}

$$H_{k+1} \equiv (H_k)' + wH_k + (k+1)\tau_k w^k H_1 + (k+1)\alpha H_k + (k+1)(n-k)\beta H_{k-1},$$

for $k = 1, 2, \dots, n-1$, (4.8)

where the τ_k as well as $\tau_0 = 1$ and $\tau_n = 0$ are given in E by

$$\tau_k = \left(\frac{1}{n^k}\right) \left(\prod_{v=1}^k (n-v)\right), \quad \text{for } k = 0, 1, \dots, n. \quad (4.9)$$

For $i, j = 0, 1, \dots, n$, polynomials $P_{i,j}$ in $y, y', \dots, y^{(n-1)}$ over E are recursively defined by

$$P_{i,j} \equiv 0, \quad \text{for } i = 0, 1, \dots, n \text{ and } i \leq j \leq n; \quad P_{1,0} \equiv 1; \quad (4.10)$$

$$\begin{aligned} P_{i+1,0} &\equiv y(P_{i,0})' + (-iy' + (i+1)\alpha y)P_{i,0} \\ &\quad + (i+1)(n-i)\beta y^2 P_{i-1,0} + (i+1)\tau_i(y')', \\ &\quad \text{for } i = 1, 2, \dots, n-1; \end{aligned} \quad (4.11)$$

$$\begin{aligned} P_{i+1,j} &\equiv y(P_{i,j})' + (-iy' + (i+1)\alpha y)P_{i,j} \\ &\quad + (i+1)(n-i)\beta y^2 P_{i-1,j} + yP_{i,j-1}, \\ &\quad \text{for } i = 1, 2, \dots, n-1 \text{ and } 1 \leq j \leq i. \end{aligned} \quad (4.12)$$

For $k=0$, the sum in

$$y^{k+1}H_k \equiv \sum_{j=0}^{k-1} (P_{k,j})(Q_n^{(j)}) \quad (4.13)$$

is vacuous and (4.13) reduces to $0=0$. Due to $y^2H_1 \equiv Q_n$ and $P_{1,0} \equiv 1$ from (4.7) and (4.10), we see that (4.13) is also valid for $k=1$. When $n \geq 2$, we use (4.8), (4.7), (4.9), $yw = y'$, (4.10), (4.11), and (4.12) to obtain

$$y^3H_2 \equiv y(Q_n)' + (((n-2)/n)y' + 2\alpha y)(Q_n) \equiv (P_{2,1})(Q_n)' + (P_{2,0})(Q_n).$$

Thus, when $n \geq 2$, (4.13) is valid for $k=2$. Suppose r is an integer such that $2 \leq r < n$ and (4.13) is valid for $k=0, 1, \dots, r$. Then, using (4.8) for $k=r$ as well as

$$y^{r+1}(H_r)' \equiv (y^{r+1}H_r)' - (r+1)y'y'H_r,$$

$yw = y'$, (4.13) for $k=r$, (4.13) for $k=r-1$, (4.7), (4.11), (4.12), and (4.10), we obtain

$$\begin{aligned} y^{r+2}H_{r+1} &\equiv y(y^{r+1}H_r)' + (-ry^{r+1}y' + (r+1)\alpha y^{r+2})(H_r) \\ &\quad + (r+1)(n-r)\beta y^{r+2}H_{r-1} + (r+1)\tau_r(y')'y^2H_1 \\ &\equiv \sum_{j=0}^{r-1} y(P_{r,j})'Q_n^{(j)} + \sum_{j=0}^{r-1} yP_{r,j}Q_n^{(j+1)} \\ &\quad + \sum_{j=0}^{r-1} (-ry' + (r+1)\alpha y)P_{r,j}Q_n^{(j)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{r-2} (r+1)(n-r) \beta y^2 P_{r-1,j} Q_n^{(j)} + (r+1)(\tau_r)(y')^r Q_n \\
& \equiv (P_{r+1,0})(Q_n) + \sum_{j=1}^{r-2} (P_{r+1,j})(Q_n^{(j)}) + (y)(P_{r,r-1})(Q_n^{(r)}) \\
& \quad + (y(P_{r,r-1})' + yP_{r,r-2} + (-ry' + (r+1)\alpha y)(P_{r,r-1}))(Q_n^{(r-1)}) \\
& \equiv \sum_{j=0}^r (P_{r+1,j})(Q_n^{(j)}).
\end{aligned}$$

Thus, (4.13) is valid for $k=0, 1, \dots, n$.

Using (4.7), (4.1), $y'/y = w$, (4.6), and (4.9), we deduce

$$H_1 \equiv \frac{y''}{y} + \alpha \frac{y'}{y} + n\beta - \left(\frac{n-1}{n}\right) w^2 \equiv V_2 + \alpha V_1 + n\beta V_0 - \tau_1 w^2. \quad (4.14)$$

Therefore, due to $a_{1,2}=1$, $a_{1,1}=\alpha$, $a_{1,0}=n\beta$, and (4.14), we see that the formula

$$H_k \equiv \left(\sum_{j=0}^{k+1} (a_{k,j})(V_j) \right) - \tau_k w^{k+1} \quad (4.15)$$

is valid for $k=1$. Also, (4.15) is valid for $k=0$. Suppose r is a positive integer such that $1 \leq r < n$ and (4.15) is valid for $k=0, 1, \dots, r$. Of course, (4.14) and (4.5) yield

$$H_1 \equiv w' + w^2 + \alpha w + n\beta - \tau_1 w^2. \quad (4.16)$$

We use (4.8), (4.15) for $k=r$, (4.15) for $k=r-1$, and (4.16) to obtain

$$\begin{aligned}
H_{r+1} & \equiv \sum_{j=0}^{r+1} (a_{r,j})' (V_j) + \sum_{j=0}^{r+1} (a_{r,j})((V_j)' + wV_j) \\
& \quad + \sum_{j=0}^{r+1} (r+1)\alpha a_{r,j} V_j + \sum_{j=0}^r (r+1)(n-r)\beta a_{r-1,j} V_j + T, \quad (4.17)
\end{aligned}$$

where

$$\begin{aligned}
T & \equiv -(r+1)\tau_r w^r w' - \tau_r w^{r+2} + (r+1)\tau_r w^r (w' + w^2 + \alpha w + n\beta - \tau_1 w^2) \\
& \quad - (r+1)\alpha \tau_r w^{r+1} - (r+1)(n-r)\beta \tau_{r-1} w^r.
\end{aligned}$$

Referring to (4.9), we find $n\tau_r - (n-r)\tau_{r-1} = 0$ and

$$T \equiv (r - (r+1)\tau_1)\tau_r w^{r+2} \equiv -\tau_{r+1} w^{r+2}. \quad (4.18)$$

We use (4.17), (4.5), $a_{r,-1}=0$, $a_{r,r+1}=1=a_{r+1,r+2}$, $a_{r-1,r+1}=0$, (4.4), and (4.18) to obtain

$$H_{r+1} \equiv \left(\sum_{j=0}^{r+2} (a_{r+1,j})(V_j) \right) - \tau_{r+1} w^{r+2}.$$

Thus, (4.15) is valid for $k=0, 1, \dots, n$.

Noting that $\tau_n=0$, we set $k=n$ in (4.13) and (4.15) to deduce

$$\sum_{j=0}^{n-1} (P_{n,j})(Q_n^{(j)}) \equiv y^{n+1} H_n \equiv y^n \left(\sum_{j=0}^{n+1} (a_{n,j})(yV_j) \right).$$

This with $yV_j = y^{(j)}$ and $a_{n,n+1}=1$ yields (4.3).

We obtain $P_{1,0} \equiv 1$, $P_{2,1} \equiv y$, ..., $P_{n,n-1} \equiv y^{n-1}$ from (4.10) and (4.12). Suppose $n \geq 2$. Then, (4.10), (4.11), and (4.12) show that, for $i=2, \dots, n$, the polynomials

$$P_{i,0}, P_{i+1,1}, \dots, P_{n,n-i}$$

in y, y', y'', \dots over E do not involve $y^{(i)}, y^{(i+1)}, \dots$. Thus, the notation in (4.2) is appropriate. This completes the proof.

THEOREM 4.2. *In addition to the hypotheses of Theorem 4.1, suppose the characteristic of E is zero and Q_n in (4.1) satisfies an identity*

$$\sum_{v=0}^{\mu} (R_v)(Q_n^{(v)}) \equiv y^k \left(y^{(n+1)} + \sum_{j=1}^{n+1} \eta_{n+1,j} y^{(n+1-j)} \right), \quad (4.19)$$

for some integers $\mu \geq 0$ and $k \geq 1$, some polynomials R_0, \dots, R_μ in y, y', \dots over E , and some elements $\eta_{n+1,1}, \dots, \eta_{n+1,n+1}$ in E . Then, these elements are unique and given in terms of (4.4) by

$$\eta_{n+1,j} = a_{n,n+1-j}, \quad \text{for } j=1, 2, \dots, n+1. \quad (4.20)$$

Moreover, there is an extension field F of E and a derivation $'$ of F whose restriction to E is the given derivation of E such that: (i) there are two solutions in F of

$$u'' + \alpha u' + \beta u = 0 \quad (4.21)$$

that are linearly independent over the subfield \mathcal{C}_F of constants in F ; (ii) if ϕ, ψ are any two solutions in F of (4.21) that are linearly independent over \mathcal{C}_F , then the set

$$\{\phi^{n-i}\psi^i: \text{for } i=0, 1, \dots, n\} \quad (4.22)$$

is linearly independent over \mathcal{C}_F and the solutions in F of

$$y^{(n+1)} + \sum_{j=1}^{n+1} \eta_{n+1-j} y^{(n+1-j)} = 0 \quad (4.23)$$

are the linear combinations over \mathcal{C}_F of (4.22); and, (iii) for any solution u_0 in F of (4.21), $y_0 = (u_0)^n$ is a solution of (4.23).

Proof. We refer to [31, p. 27] for the existence of an extension field F of E and a derivation $'$ of F , whose restriction to E is the given derivation of E , such that there are two solutions in F of (4.21) linearly independent over \mathcal{C}_F .

We apply to Q_n in (4.1) the substitution

$$y = u^n, \quad y' = nu^{n-1}u', \quad y'' = nu^{n-1}u'' + n(n-1)u^{n-2}(u')^2,$$

used in [39, p. 35], to obtain

$$Q_n \equiv nu^{2n-1}(u'' + \alpha u' + \beta u).$$

Thus, for any u_0 in F , the element $y_0 = (u_0)^n$ is a solution of $Q_n = 0$ if and only if u_0 is a solution of (4.21). Due to (4.19) and (4.3), we see that $y_0 = (u_0)^n$ is a solution of both (4.23) and

$$y^{(n+1)} + \sum_{j=1}^{n+1} a_{n,n+1-j} y^{(n+1-j)} = 0 \quad (4.24)$$

whenever u_0 is a solution of (4.21).

Let ϕ, ψ be solutions in F of (4.21) that are linearly independent over \mathcal{C}_F . Then, the elements

$$y_j = (\phi + j\psi)^n = \sum_{i=0}^n \binom{n}{i} j^i \phi^{n-i} \psi^i, \quad \text{for } j = 0, 1, \dots, n,$$

are solutions of both (4.23) and (4.24). Agreeing that $0^0 = 1$, setting

$$h_{i,j} = \binom{n}{i} j^i, \quad \text{for } i, j = 0, 1, \dots, n,$$

and introducing the $(n+1) \times (n+1)$ matrix $H = [h_{i,j}]$, we relate $\det H$ to the determinant of a Vandermonde matrix to obtain

$$\det H = (n!) \left(\prod_{i=1}^n \binom{n}{i} \right) \left(\prod_{1 \leq i < j \leq n} (j-i) \right) = \prod_{k=1}^n k^k \neq 0. \quad (4.25)$$

Thus, each of $\phi^{n-i}\psi^i$, for $i=0, 1, \dots, n$, is a linear combination over \mathcal{C}_F of y_0, y_1, \dots, y_n and is therefore a solution of both (4.23) and (4.24). Proposition 5.2 shows that (4.22) is linearly independent over \mathcal{C}_F . Thus, the solutions in F of (4.23) and the solutions in F of (4.24) are the linear combinations over \mathcal{C}_F of (4.22). Subtracting (4.24) from (4.23), we find that the equation

$$\sum_{j=1}^{n+1} (\eta_{n+1,j} - a_{n,n+1-j}) y^{(n+1-j)} = 0$$

has (4.22) as a set of $n+1$ linearly independent solutions. This yields (4.20) and completes the proof.

Proposition 3.6 motivated our discovery of Theorems 4.1 and 4.2. We first found relations like (4.19) for $n=1, 2, 3, 4, 5$ by solving linear algebraic equations for the coefficients of the polynomials R_v that produce the indicated factorization. This showed that, *when k is minimal, the polynomials R_v for (4.19) are unique if and only if $1 \leq n \leq 4$* . When $n=1$ or $n=2$, minimal k for (4.19) is $k=1$ and the R_v for $k=1$ are unique elements in E . When $n=3$ or $n=4$, minimal k for (4.19) is $k=2$ and the R_v for $k=2$ are unique homogeneous linear polynomials in y, y', \dots over E . When $n \geq 5$, minimal k for (4.19) exceeds 2 and the R_v are nonlinear polynomials in y, y', \dots over E that are not unique. In particular, minimal k in (4.19) for $n=5$ is $k=3$ and addition of $Q_5 Q_5^{(1)} - Q_5^{(1)} Q_5$ to the left member of any (4.19) having $n=5$ and $k=3$ yields a different identity of the same type.

PROPOSITION 4.3. *Suppose the characteristic of E in Theorem 4.1 is not two or three. Then, (4.4) yields*

$$a_{n,n} = \binom{n+1}{2} \alpha, \quad (4.26)$$

$$a_{n,n-1} = \binom{n+1}{3} \alpha' + \frac{3n+2}{4} \binom{n+1}{3} \alpha^2 + \binom{n+2}{3} \beta, \quad (4.27)$$

and, when $n \geq 2$,

$$\begin{aligned} a_{n,n-2} = & \binom{n+1}{4} \alpha'' + (2n+1) \binom{n+1}{4} \alpha \alpha' + 2 \binom{n+2}{4} \beta' \\ & + \binom{n+1}{2} \binom{n+1}{4} \alpha^3 + 2n \binom{n+2}{4} \alpha \beta. \end{aligned} \quad (4.28)$$

Proof. Using (4.4), we find

$$a_{k,k} = \sum_{i=0}^{k-1} (a_{i+1,i+1} - a_{i,i}) = \sum_{i=0}^{k-1} (i+1) \alpha = \binom{k+1}{2} \alpha, \quad (4.29)$$

for $k = 0, 1, \dots, n$. This yields (4.26). We use (4.4), (4.29), and the calculus of finite differences to deduce

$$\begin{aligned} a_{k+1,k} &= \sum_{i=0}^k (a_{i+1,i} - a_{i,i-1}) \\ &= n\beta + \sum_{i=1}^k ((a_{i,i})' + (i+1)\alpha a_{i,i} + (i+1)(n-i)\beta) \\ &= \binom{k+2}{3} \alpha' + \frac{3k+5}{4} \binom{k+2}{3} \alpha^2 + \frac{3n-2k}{3} \binom{k+2}{2} \beta, \quad (4.30) \end{aligned}$$

for $k = 0, \dots, n-1$. This gives (4.27). Suppose $n \geq 2$. Then, application of (4.4), (4.29), (4.30), and the calculus of finite differences yields

$$\begin{aligned} a_{k+2,k} &= \sum_{i=0}^k (a_{i+2,i} - a_{i+1,i-1}) \\ &= \sum_{i=0}^k ((a_{i+1,i})' + (i+2)\alpha a_{i+1,i} + (i+2)(n-i-1)\beta a_{i,i}) \\ &= \binom{k+3}{4} \alpha'' + (2k+5) \binom{k+3}{4} \alpha \alpha' + \frac{2n-k}{2} \binom{k+3}{3} \beta' \\ &\quad + \binom{k+3}{2} \binom{k+3}{4} \alpha^3 + \frac{3nk+4n-2k^2-4k}{2} \binom{k+3}{3} \alpha \beta, \end{aligned}$$

for $k = 0, \dots, n-2$. We set $k = n-2$ in this to obtain (4.28). This completes the proof.

PROPOSITION 4.4. *For $c_1(z)$, $c_2(z)$ in E_Ω and $m \geq 2$, the meromorphic functions $\xi_{m,1}(z)$, ..., $\xi_{m,m}(z)$ in E_Ω defined by (1.12), (1.13), (1.14), and (1.11) satisfy (1.8) and the requirement that (1.9) has a fundamental system of local solutions of the form (1.10) on some subregion of Ω .*

Proof. We apply Theorems 4.1 and 4.2 to the situation where E is E_Ω , $n = m-1$, and α, β in E are the $\alpha(z)$, $\beta(z)$ in E_Ω defined by (1.12), (1.13). Then, (4.4) reduces to (1.14) and the elements (1.11) are the coefficients (4.20) of (4.23) corresponding to (4.3). There are two linearly independent analytic solutions $\phi(z)$, $\psi(z)$ of (4.21) on some subregion U of Ω . Therefore, the proof of Theorem 4.2 shows that (4.23), which now coincides with (1.9), has (4.22), or equivalently (1.10), as a fundamental system of solutions on U .

We set $n = m-1$ in (4.26) and use (1.12) to deduce

$$\xi_{m,1}(z) \equiv a_{m-1,m-1}(z) \equiv \binom{m}{2} \alpha(z) \equiv c_1(z).$$

Setting $n = m - 1$ in (4.27), we use (1.12) and (1.13) to obtain

$$\begin{aligned}\xi_{m,2}(z) &\equiv a_{m-1,m-2}(z) \equiv \binom{m}{3} \alpha'(z) \\ &\quad + \frac{3m-1}{4} \binom{m}{3} (\alpha(z))^2 + \binom{m+1}{3} \beta(z) \\ &\equiv c_2(z).\end{aligned}$$

This yields (1.8) and completes the proof.

For any positive integer n and each $\alpha(z)$, $\beta(z)$ in E_Ω , there are unique elements $a_{n,0}(z), \dots, a_{n,n}(z)$ in E_Ω such that the n th power $y(z) \equiv (u(z))^n$ of each local solution $u(z)$ of

$$u'' + \alpha(z)u' + \beta(z)u = 0 \quad (4.31)$$

is a local solution of $L_n = 0$, where

$$L_n \equiv y^{(n+1)} + \sum_{j=0}^n a_{n,j}(z) y^{(j)}. \quad (4.32)$$

The important problem of specifying $a_{n,0}(z), \dots, a_{n,n}(z)$ explicitly in terms of $\alpha(z)$, $\beta(z)$, and n was considered in [34, pp. 429–431; 20, pp. 129–131; 2, pp. 212–213; 3, pp. 411–414; 21, p. 705; 22, p. 333; 24, p. 169; 42, p. 22; 28, p. 54; 41]. Our new method based on (4.4) is far more effective than any of the earlier procedures.

EXAMPLE 4.5. We apply (4.4) to deduce

$$\begin{aligned}a_{2,0} &\equiv n\beta' + 2n\alpha\beta, & a_{2,1} &\equiv \alpha' + 2\alpha^2 + (3n-2)\beta, & a_{2,2} &\equiv 3\alpha, \\ a_{3,0} &\equiv n\beta'' + 2n\alpha'\beta + 5n\alpha\beta' + 6n\alpha^2\beta + 3(n-2)n\beta^2, \\ a_{3,1} &\equiv \alpha'' + 7\alpha\alpha' + 6\alpha^3 + (14n-12)\alpha\beta + (4n-2)\beta', \\ a_{3,2} &\equiv 4\alpha' + 11\alpha^2 + (6n-8)\beta, & a_{3,3} &\equiv 6\alpha.\end{aligned}$$

Setting $n = 2$, we see that L_2 is given by (3.24). For $n = 3$, we obtain

$$\begin{aligned}L_3 &\equiv y^{(4)} + 6\alpha y''' + (4\alpha' + 11\alpha^2 + 10\beta) y'' \\ &\quad + (\alpha'' + 7\alpha\alpha' + 6\alpha^3 + 30\alpha\beta + 10\beta') y' \\ &\quad + (3\beta'' + 6\alpha'\beta + 15\alpha\beta' + 18\alpha^2\beta + 9\beta^2) y.\end{aligned}$$

Of course, L_1 is $y'' + \alpha y' + \beta y$.

The research in [40, 4, 46] establishes that, for any monic homogeneous

linear differential equations $M=0$ and $N=0$ over E_Ω , there is a unique monic homogeneous linear differential equation $P=0$ of least order over E_Ω such that $y_1(z)y_2(z)$ is a local solution of $P=0$ whenever $y_1(z)$ is a solution of $M=0$ on some subregion U of Ω and $y_2(z)$ is a solution of $N=0$ on U . In terms of fixed $\alpha(z)$, $\beta(z)$ from E_Ω , let L_n be defined by (4.32) and (4.4) for $n=1, 2, 3, \dots$. We can easily verify that, for any $i, j \geq 1$, if $M \equiv L_i$ and $N \equiv L_j$, then $P \equiv L_{i+j}$. In particular, if $M \equiv L_n$ and $N \equiv L_1$, then $P \equiv L_{n+1}$. Of course, (4.4) does not relate directly the coefficients of L_{n+1} to those of L_n and L_1 . Instead, it suggests the difficulty of the general problem in [46] of algorithmically expressing the coefficients for $P=0$ in terms of those for $M=0$ and $N=0$.

5. SEVERAL WRONSKIANs

We assume F is a field of characteristic 0 having a derivation $'$ and we let \mathcal{C}_F denote its subfield of constants. Given any f, f_1, f_2, \dots, f_n in F , we have

$$W(ff_1, ff_2, \dots, ff_n) = f^n W(f_1, f_2, \dots, f_n) \quad (5.1)$$

for the indicated Wronskians relative to $'$. Directly applicable to establish (5.1) are the proofs in [19, p. 276; 35, pp. 662–663] for the original formulation of [12, p. 298].

THEOREM 5.1. *Suppose ϕ, ψ are elements of F that are linearly independent over \mathcal{C}_F , let m be any integer, let n be a positive integer, and set*

$$y_j = (\phi + j\psi)^m, \quad \text{for } j=0, 1, \dots, n.$$

Then, the Wronskian $\Gamma_{m,n} = W(y_0, y_1, \dots, y_n)$ is given by

$$\Gamma_{m,n} = \left(\prod_{k=0}^n ((k!)(m-n+k)^k (\phi + k\psi)^{m-n}) \right) (\phi\psi' - \phi'\psi)^{n(n+1)/2}, \quad (5.2)$$

where $0! = 0^0 = 1$. Moreover, $\Gamma_{m,n} \neq 0$, when either $m < 0$ or $m \geq n$; and $\Gamma_{m,n} = 0$ when $0 \leq m < n$.

Proof. Setting $\chi = \psi/\phi$, we use (5.1) to obtain

$$\begin{aligned} \Gamma_{m,n} &= \phi^{m(n+1)} W(1, (1+\chi)^m, (1+2\chi)^m, \dots, (1+n\chi)^m) \\ &= (\phi^{m(n+1)})(\det[a_{i,j}]), \end{aligned}$$

where $a_{i,j} = (mj(1+j\chi)^{m-1}\chi')^{(i-1)}$, for $i, j=1, 2, \dots, n$. Starting with the $n \times n$ matrix $[a_{i,j}]$, we apply elementary row operations of the type where to one row is added a multiple from F of another row. When $0 \leq m < n$, a

succession of such operations leads to a matrix having a row of zeros so that $\Gamma_{m,n} = 0$. And, the right member of (5.2) is also zero because it has the factor $(m - n + k)^k$ for $k = n - m$.

Suppose that either $m < 0$ or $m \geq n$. Starting with $[a_{i,j}]$, we apply a succession of elementary row operations of the kind mentioned above to obtain the $n \times n$ matrix $[b_{i,j}]$ having

$$b_{i,j} = m(m-1) \cdots (m-i+1) j^i (1+j\chi)^{m-i} (\chi')^i, \\ \text{for } i, j = 1, 2, \dots, n,$$

and $\det[a_{i,j}] = \det[b_{i,j}]$. This yields

$$\Gamma_{m,n} = \phi^{m(n+1)} \left(\prod_{k=1}^n ((m-n+k)^k (1+k\chi)^{m-n}) \right) \\ \times ((\chi')^{n(n+1)/2}) (\det[c_{i,j}]), \quad (5.3)$$

where $[c_{i,j}]$ is the $n \times n$ matrix having

$$c_{i,j} = j^i (1+j\chi)^{n-i}, \quad \text{for } i, j = 1, 2, \dots, n.$$

We observe that $\det[c_{i,j}] = (n!)^n (\det[d_{i,j}])$, where

$$d_{i,j} = (c_{i,j})/(j^n) = ((1/j) + \chi)^{n-i}, \quad \text{for } i, j = 1, 2, \dots, n.$$

Since $[d_{i,j}]$ is a Vandermonde matrix, we find $\det[c_{i,j}] = P_n$, where $P_1 = 1$ and

$$P_k = (k!)^k \left(\prod_{1 \leq i < j \leq k} ((1/i) - (1/j)) \right), \quad \text{for } k = 2, 3, \dots$$

This gives $P_{k+1}/P_k = (k+1)!$, for $k = 1, 2, \dots$, and $P_n = (1!)(2!) \cdots (n!)$. We use this with (5.3) to obtain (5.2), $\Gamma_{m,n} \neq 0$, and complete the proof.

COROLLARY 5.2. *Given any ϕ, ψ in F and any positive integer n , the Wronskian $\Gamma = W(\phi^n, \phi^{n-1}\psi, \dots, \psi^n)$ of $\phi^{n-i}\psi^i$, for $i = 0, 1, \dots, n$, satisfies*

$$\Gamma = \left(\prod_{k=1}^n (k!) \right) (\phi\psi' - \phi'\psi)^{n(n+1)/2}. \quad (5.4)$$

Proof. If ϕ, ψ are linearly dependent over \mathcal{C}_F , then (4.22) is linearly dependent over \mathcal{C}_F and (5.4) reduces to $0 = 0$. Suppose ϕ, ψ are linearly independent over \mathcal{C}_F and set

$$y_j = (\phi + j\psi)^n = \sum_{k=0}^n \binom{n}{k} j^k \phi^{n-k} \psi^k, \quad \text{for } j = 0, 1, \dots, n. \quad (5.5)$$

Let $M = [f_{i,j}]$, $N = [g_{i,j}]$, $H = [h_{i,j}]$ denote the matrices defined by

$$f_{i,j} = (y_j)^{(i)}, \quad g_{i,j} = (\phi^{m-j}\psi^j)^{(i)}, \quad h_{i,j} = \binom{n}{i} j^i,$$

for $i, j = 0, 1, \dots, n$. Using (5.5) and the notation of Theorem 5.1 for $m = n$, we obtain $M = NH$ and

$$\Gamma_{n,n} = \det M = (\det N)(\det H) = (\det H)\Gamma. \quad (5.6)$$

We combine (5.6), (4.25), and (5.2) for $m = n$ to deduce (5.4) and complete the proof.

Special cases of Corollary 5.2 have appeared in [28, p. 32; 17, pp. 260–262; 47, pp. 278–279].

PROPOSITION 5.3. *Suppose $f(z)$, $g(z)$ are analytic functions on a region U of the complex plane and C_0, C_1, \dots are complex numbers. Then, for any positive integer n , the Wronskian*

$$W_n \equiv W(\exp(C_0 f + g), \exp(C_1 f + g), \dots, \exp(C_n f + g))$$

is given by

$$W_n \equiv \left(\prod_{0 \leq i < j \leq n} (C_j - C_i) \right) \left(\prod_{k=0}^n \exp(C_k f + g) \right) (f')^{n(n+1)/2}. \quad (5.7)$$

Proof. Several applications of (5.1) yield

$$\begin{aligned} W_{n+1} &\equiv W(e^{(C_0 f + g)}, \dots, e^{(C_n f + g)}, e^{(C_{n+1} f + g)}) \\ &\equiv (e^{(C_{n+1} f + g)})^{n+2} W(e^{(C_0 - C_{n+1}) f}, \dots, e^{(C_n - C_{n+1}) f}, 1) \\ &\equiv (e^{(C_{n+1} f + g)})^{n+2} (-1)^{n+1} \gamma_n, \end{aligned}$$

where

$$\begin{aligned} \gamma_n &\equiv W((C_0 - C_{n+1}) f' e^{(C_0 - C_{n+1}) f}, \dots, (C_n - C_{n+1}) f' e^{(C_n - C_{n+1}) f}) \\ &\equiv \left(\prod_{k=0}^n (C_k - C_{n+1}) \right) (f')^{n+1} W(e^{(C_0 - C_{n+1}) f}, \dots, e^{(C_n - C_{n+1}) f}). \end{aligned}$$

Using (5.1) again, we obtain

$$W_{n+1} \equiv (e^{(C_{n+1} f + g)}) \left(\prod_{k=0}^n (C_{n+1} - C_k) \right) (f')^{n+1} W_n. \quad (5.8)$$

Formula (5.7) is correct for $n = 1$. And, (5.8) shows that if (5.7) is correct for some positive integer, then it is correct for the next greater integer. Thus, (5.7) is valid for $n = 1, 2, \dots$. This completes the proof.

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